Analytic Travelling Wave Solutions and Numerical Analysis of Fisher's Equation via Explicit-Implicit FDM

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Original Research Article

ABSTRACT

We study the nonlinear parabolic Fisher's equations for travelling wave solutions. The analyses focus on to describe the analytic solution in the spatial pattern of travelling wave solutions; especially the solutions are characterized in invariant with respect to translation in space. There are two phases in the work: in the first stage, we analyze dimensional reaction-diffusion equation with logistic type growth while in the second phase the non-dimensional equation known as Fishers' equation is studied numerically. To investigate the results numerically, we select the explicit-implicit finite difference method (FDM) and the approximate solutions are compared with the exact solution in different time steps.

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1. INTRODUCTION

Various types of natural processes which entail mechanisms through reaction-diffusion equations and one of the most important examples of nonlinear reaction-diffusion equation are Fisher's equation. This equation has been used for designating several types of physical framework like heat and mass transfer, flame propagation, chemical reactions and so on. In gene technology, the necessity of Fisher's equation is discussed via travelling wave solutions and has been studied in the propagation of a gene within a population [1]. Ronald Fisher presented this model in [1] and his paper consisted of population dynamics to describe the spatial spread of an advantageous allele. The Russian mathematician (1903-1987), Andrey Nikolaevich Kolmogorov took a part on this equation also known as Kolmogorov-Petrovsky-Piskunov (KPP) and widely familiar as Fisher-KPP equation [2].

In literature, several studies are available and many researchers worked on this equation, see for example in [1-7] and references therein. An analytic method to construct explicitly exact and approximate solutions for nonlinear evolution equations is suggested by Feng [3,4]. These solutions included solitary wave solutions, singular traveling wave solutions, and periodical wave solutions. As a continuation of the previous study, Demina studied the meromorphic solutions (including rational, periodic, elliptic) of autonomous nonlinear ordinary differential equations and developed an algorithm for constructing meromorphic solutions. Next Yuan in [5] introduced the complex method for solving nonlinear Fisher's Kolmogorov equation of degree three. Tyson and Brazhnik discussed about travelling wave solution for this type of nonlinear equation in two spatial dimensions [6]. A numerical scheme to solve this equation was developed by Tang and Weber [7]. George Adomian introduced another powerful technique known as Adomian decomposition method (ADM) [8] which is useful for solving nonlinear problem like fisher's equation. Fisher's equation is one of the simplest semi-linear reaction diffusion equations and it can exhibit traveling wave solution that switch between equilibrium states. To execute the behavior of neutron population in a nuclear reactor, Canosa [9,10] used a particular case of the equation which was introduced in [1]. Further, Haar wavelet was utilized by Hariharan et al. [11]. Ablowitz and Zepetella [12] used Laurent series expansion to solve Fisher's equation. Since travelling wave plays an important role in biology, Murray's authoritative work 'Mathematical Biology' was dedicated to biological waves [13]. To get the Fisher's equation and it's travelling wave solutions, the platform was ordinary differential equations [14,15] and for numerical study and error analysis, the literature [16,17] can be studied. However, several robust numerical techniques were presented in [18-28] to solve various types of non-linear partial differential equations.

In this paper, we consider Fisher's equation to analyze both analytically and numerically using implicit-explicit finite difference methods. This study has the following novelties:

- Theoretical studies are verified by numerical simulations using Implicit-Explicit Finite Difference Method.
- The polynomial fit data match with the numerical approximations and there is a better agreement with the exact solutions; see Fig. 4(b), 5(b) and in Fig. 7.
- The convergence analysis and stability of the numerical method are presented in a new approach known as the primitive variable transformation.

The paper is organized as follows: in Section 2, we consider the general Fisher's equation connected with logistic growth function while all the results are presented in Section 3. Continually we translated the equation into a dimensionless form in Sub-section 3.1. In the next Sub-section 3.2, we explore the travelling wave solutions of a special case of Fisher's equation analytically. The numerical solutions are presented graphically to validate the theoretical results in Sub-section 3.3 while comparing with the exact solution. The stability analysis of the equilibrium points are studied in Section 3.4 and describe the error analysis. The convergence analyses of numerical method are discussed in Section 4. Finally, Section 5 concludes the summary and conclusion of the paper.

2. FISHER'S EQUATION

Let us consider the following reaction-diffusion equation in the general form

$$\frac{\partial p}{\partial t} = d\frac{\partial^2 p}{\partial x^2} + g(p) \qquad (1.1)$$

Where,

g is a nonlinear function of p and p is described as a population of organisms, particles of

chemicals, insect population, population density, or a colonial bacteria. By considering the logistic type of reaction term, the Fisher's equation now can be written in the form of

$$\frac{\partial p}{\partial t} = \mathrm{d}\frac{\partial^2 p}{\partial x^2} + rp\left(1 - \frac{p}{k}\right) \qquad (1.2)$$

Here d is the diffusion coefficient or constant, r is the intrinsic growth rate, k is the carrying capacity, t is time, x is the spatial location and $p = p(t, x)$ is the state variable of the diffusive population in location x and at time t while the reaction term follows the logistic law.

3. ANALYTIC RESULTS WITH NUMERICAL APPROXIMATIONS

At the initial stage of this section, we translated the governing equation (1.2) into the dimensionless form. Then the non-linear equation is solved analytically to get the travelling wave solutions and finally we study the problem numerically to compare with the exact solution.

3.1 Dimensional Analysis

To acquire the dimensionless form of Fisher's equation, at first we consider the dimensionless variables

$$u = \frac{p}{M} \text{ and } l = \frac{t}{N} \qquad (1.3)$$

Where, M and N are scaling parameters. Applying chain rule formula, we get

$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial l} \qquad (1.4)$$

Taking the values of u and l from equation (1.3) and using in equation (1.4), we obtain

$$\frac{\partial u}{\partial l} = \frac{N}{M}\frac{\partial p}{\partial t} \qquad (1.5)$$

Now we can re-write the equation (1.5) such that

$$\frac{\partial p}{\partial t} = \frac{M}{N}\frac{\partial u}{\partial l} \qquad (1.6)$$

Again,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$
$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{M}\frac{\partial p}{\partial t}\frac{\partial t}{\partial x}$$
$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{M}\frac{\partial p}{\partial x}$$

Then we can write

$$\frac{\partial p}{\partial x} = M\frac{\partial u}{\partial x} \qquad (1.7)$$

And also

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{M}\frac{\partial p}{\partial x}\right) = \frac{1}{M}\frac{\partial^2 p}{\partial x^2}$$

Hence we can get,

$$\frac{\partial^2 p}{\partial x^2} = M\frac{\partial^2 u}{\partial x^2} \qquad (1.8)$$

Using the relations as developed in (1.3), (1.6) and (1.8), the equation (1.2) yields

$$\frac{M}{N}\frac{\partial u}{\partial l} = \mathrm{d}M\frac{\partial^2 u}{\partial x^2} + ruM\left(1 - \frac{uM}{k}\right)$$
$$\Rightarrow \frac{\partial u}{\partial l} = \mathrm{d}N\frac{\partial^2 u}{\partial x^2} + \frac{N}{M}ruM\left(1 - \frac{uM}{k}\right)$$
$$\Rightarrow \frac{\partial u}{\partial l} = \mathrm{d}N\frac{\partial^2 u}{\partial x^2} + Nru\left(1 - \frac{u}{k/M}\right)$$

The relation $\frac{k}{M} = 1$, $Nr = 1$ implies that

$$N = \frac{1}{r} \text{ and } M = k$$

So we can say that N is the reciprocal of the intrinsic growth rate and M is the carrying capacity. After setting $\frac{k}{M} = 1$, $Nr = a$ and $\mathrm{d}N = W$ the Fisher's equation is presented in a new form such that

$$\frac{\partial u}{\partial l} = W\frac{\partial^2 u}{\partial x^2} + au(1 - u) \qquad (1.9)$$

Where,

a is the reactive factor and W is a diffusion constant.

Let us now suppose that

$$l^* = al \text{ and } x^* = x\left(\frac{a}{W}\right)^{\frac{1}{2}}$$

and rearrange the equations such that

$$l = \left(\frac{1}{a}\right)l^* \text{ and } x = x^*\left(\frac{W}{a}\right)^{\frac{1}{2}}$$

The non-dimensionalized variables give us

$$\frac{\partial u}{\partial l} = a\frac{\partial u}{\partial l^*}$$

$$\frac{\partial u}{\partial x} = \left(\frac{a}{W}\right)^{\frac{1}{2}}\frac{\partial u}{\partial x^*}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[\left(\frac{a}{W}\right)^{\frac{1}{2}}\frac{\partial u}{\partial x^*}\right]$$

$$= \left(\frac{a}{W}\right)^{\frac{1}{2}}\frac{\partial}{\partial x^*}\left[\left(\frac{a}{W}\right)^{\frac{1}{2}}\frac{\partial u}{\partial x^*}\right]$$

$$= \frac{a}{W}\frac{\partial^2 u}{\partial x^{*2}}$$

This additionally yields

$$a\frac{\partial u}{\partial l^*} = au(1-u) + W\left(\frac{a}{W}\right)\frac{\partial^2 u}{\partial x^{*2}}$$

$$\Rightarrow \frac{\partial u}{\partial l^*} = u(1-u) + \frac{\partial^2 u}{\partial x^{*2}}$$

For ignoring the superscript star "*" notation and let $l^* = t$ and $x^* = x$, we find the required dimensionless form of Fisher's equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)$$

which introduce a mutation occurring in a species distributed in a linear habitat. In this equation, $u = u(t,x)$ is density of population, x is the spatial variable and t is the time.

3.2 Solution and Exploration of Fisher's Equation

For searching the solution and exploration of Fisher's equation, we use phase portrait to describe the behavior of the roots and also use Implicit-Explicit Finite Difference Method to solve the problem. At first we discuss the travelling wave solution of the problem while the dimensionless form of the Fisher's equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)$$

Let us consider a particular case of this equation, see, for example in [14]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6u(1-u) \qquad (1.10)$$

Now we have to search for wave solution of this equation. At first we assume a wave transformation in the following form

$$u(t,x) = G(s), \qquad s = x - ct \qquad (1.11)$$

At $s \to \pm\infty$ the function G approaches to the constant values. The function G to be determined by differentiates twice. Here c is the unknown wave speed which must be figure out as a part of the solution of the problem. We have to use ordinary differential equation for finding travelling wave solution of Fisher's equation [14,15]. We can find a second order ordinary differential equation for G from (1.10) and (1.11) such that

$$-c\frac{dG}{ds} = \frac{d^2 G}{ds^2} + 6G(1-G) \qquad (1.12)$$

According to the phase plane analysis, it is necessary to analyze the equation (1.12) which cannot be solved in a closed form. In a standard way we write (1.12) as a simultaneous system of first order equations by defining $H = \frac{dG}{ds}$ and hence we obtain

$$\begin{cases} \dfrac{dG}{ds} = H \\ \dfrac{dH}{ds} = -6G(1-G) - cH \end{cases} \qquad (1.13)$$

By solving this system for equilibrium points

$$\begin{cases} 0 = H \\ 0 = -6G(1-G) - cH \end{cases} \qquad (1.14)$$

There are two stationary equilibrium points such that (0, 0) and (1, 0). The system is then linearized near the stationary points. For obtaining the eigenvalues corresponding to the equilibrium points, we have to use Jacobian matrix as well as characteristic equations. Now Jacobian matrix of the system (1.13) is

$$J(G,H) = \begin{pmatrix} 0 & 1 \\ 12G - 6 & -c \end{pmatrix}$$

At (0, 0), the matrix turn to

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -6 & -c \end{pmatrix}$$

Sequentially, we need to use the characteristic equation for finding the eigenvalues and hence define the characteristic equation as follows

$$|J - \lambda.I| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 0 & 1 \\ -6 & -c \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{matrix} -\lambda & 1 \\ -6 & -c - \lambda \end{matrix} \right| = 0$$

$$\Rightarrow (-\lambda)(-c - \lambda) + 6 = 0$$

$$\Rightarrow \lambda^2 + \lambda.c + 6 = 0$$

So the eigenvalues corresponding to the stationary point (0, 0) are

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 24}}{2}$$

Similarly, using characteristic equation, we can obtain the eigenvalues corresponding to the equilibrium point (1,0) such that

$$\lambda_{3,4} = \frac{-c \pm \sqrt{c^2 + 24}}{2}$$

We can mainly observe the behavior of the system from these roots.

- If $c \geq 2\sqrt{6}$ then $\lambda_{1,2}$ are both real and negative. Here (0, 0) is a stable node for the linearized system.
- If $c \in (0, 2\sqrt{6})$ then $\lambda_{1,2}$ are complex with negative real part and the equillibirum (0, 0) is a stable focus.
- On the other hand, $\lambda_{3,4}$ are real and opposite signs and in this case (1, 0) is a saddle point. There exists finite limits of $G(s)$ as $s \to \pm\infty$. In this situation, the equilibrium points are the limit points of solutions.

For $s \to \pm\infty$, we can find the travelling wave solutions of (1.12) which is equivalent to searching for orbits of (1.13). If they join separate equilibrium points then such orbits are known as heteroclinic orbits. If the orbit returns to the same equilibrium point from which it started known as homoclinic. There are two orbits giving rise, together with the equilibrium point (1, 0) to the unstable manifold defined at least in some neighborhood of the saddle point (1, 0) such that each orbit $\mu(s) = (G(s), H(s))$ satisfies $\mu(s)$ to (1, 0) as $s \to -\infty$. At least one of these orbits can be continued till $s \to +\infty$ and reaches then (0, 0) in a monotonic way and we can get an exact solution of equation (1.10) using initial-boundary conditions such that
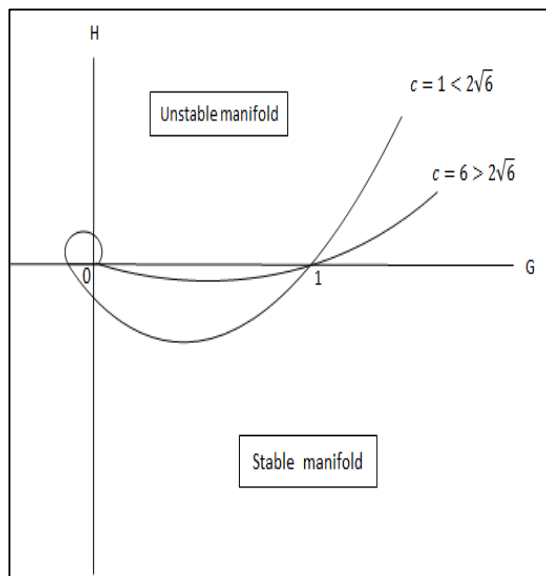
$$G(s) = u(t,x) = \frac{1}{(1 + e^{(x-5t)})^2}$$

Fig. 1. Phase portrait with c=1 and c=6

For any $c > 0$, ther exists a unique right-going travelling wave with speed c connecting the state $u = 1$, $u_s = 0$ for $x \to -\infty$ to the state $u = 0$, $u_s = 0$ for $x \to +\infty$. Then we will find faster waves.

- For $c \geq 2\sqrt{6}$, the wave monotonically decreasing function of x, while for $c < 2\sqrt{6}$, it is oscillatory.
- That is, the critical points in the $G - H$ plane are (1, 0), a saddle point and (0, 0), a stable node for $c \geq 2\sqrt{6}$ and a spiral for $c < 2\sqrt{6}$.
- So, the orbit is globally defined for $s = x - ct \in (-\infty, \infty)$ joining equilibrium points (1, 0) and (0, 0). Hence G is monotonically decreasing and becomes flat at "$\pm\infty$" giving a travelling wave front solution.

3.3 Implicit-Explicit Finite Difference Method

We use numerical methods for solving this model and comparing its approximate solution with travelling wave solutions of Fisher's equations. Various types of implicit-explicit finite difference method are important for solving nonlinear partial differential equations.

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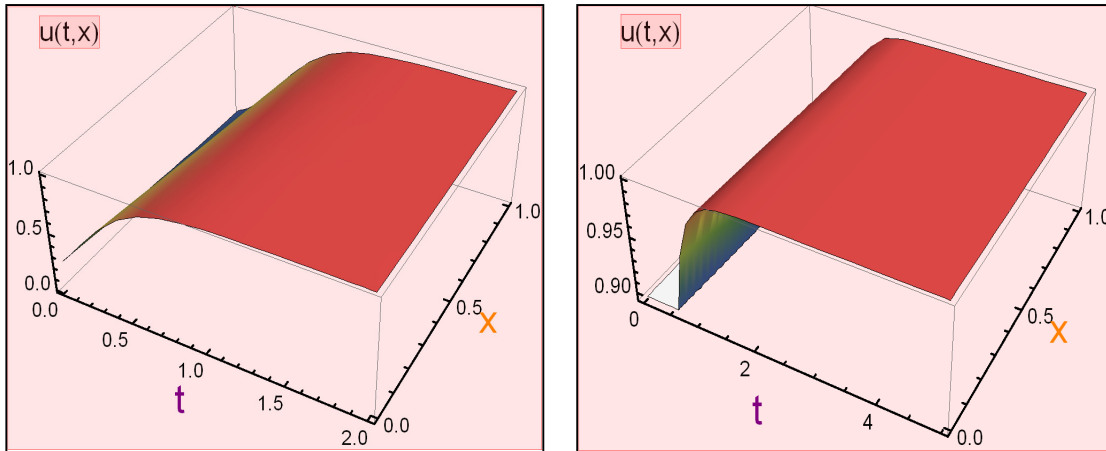
Fig. 1(a). Exact solution $G(s) = u(t, x)$ for different times over the domain

The main motivation to introduce implicit-explicit finite difference method is to compare the approximate solution with the exact one and polynomial fit data as a new dimension. Now we introduce implicit-explicit method [16] for solving the governing equations as recalled here

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + rp\left(1 - \frac{p}{k}\right) \tag{1.15}$$

Since we have obtained equation (1.10) which is a dimensionless form of the equation (1.15) as defined in the earlier section and hence we can write

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6u(1 - u) \tag{1.16}$$

Where,

the domain, $\sigma = (0,1)$
the initial condition, $u(0, x) = \frac{1}{(1 + e^x)^2}$ and
the boundary conditions,
$$u(t, 0) = \frac{1}{(1 + e^{-5t})^2}$$
$$u(t, 1) = \frac{1}{(1 + e^{1-5t})^2}$$

Obtaining the difference method of the equation (1.16), at first, we have to use the Taylor series in t to form the difference quotient

$$\frac{\partial u}{\partial t}(t_j, x_i) = \frac{u(t_j + \Delta t, x_i) - u(t_j, x_i)}{\Delta t} - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(\tau_j, x_i) \tag{1.17}$$

for some $\tau_j \in (t_j, t_{j+1})$ and $\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}(\tau_j, x_i)$ is the error term.

Now using central-difference method to form the difference quotient by Taylor series in x, we have

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2}(t_j, x_i) \\ &= \left[\frac{u(t_j, x_i + \Delta x) - 2u(t_j, x_i) + u(t_j, x_i - \Delta x)}{(\Delta x)^2}\right. \\ &\quad - \frac{(\Delta x)^2}{6}\frac{\partial^4 u}{\partial x^4}(t_j, \gamma_i) \end{aligned} \tag{1.18}$$

Where,

$\gamma_i \in (x_{i-1}, x_{i+1})$ and (t_j, x_i) is the interior gridpoint and $\frac{(\Delta x)^2}{6}\frac{\partial^4 u}{\partial x^4}(t_j, \gamma_i)$ is the error.

Suppose that, $\Delta x = h$, $\Delta t = K$. Then (1.17) becomes

$$\begin{aligned} &\frac{\partial u}{\partial t}(t_j, x_i) \\ &= \frac{u(t_j + K, x_i) - u(t_j, x_i)}{K} \\ &\quad - \frac{K}{2}\frac{\partial^2 u}{\partial t^2}(\tau_j, x_i) \end{aligned} \tag{1.19}$$

and (1.18) becomes

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2}(t_j, x_i) \\ &= \left[\frac{u(t_j, x_i + h) - 2u(t_j, x_i) + u(t_j, x_i - h)}{h^2}\right] \\ &\quad - \frac{h^2}{6}\frac{\partial^4 u}{\partial x^4}(t_j, \gamma_i) \end{aligned} \tag{1.20}$$

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Putting (1.19) and (1.20) in (1.16) and ignoring the local truncation error of order $O(K + h^2)$ consisting of $-\frac{K}{2}\frac{\partial^2 u}{\partial t^2}(\tau_j, x_i)$ and $-\frac{h^2}{6}\frac{\partial^4 u}{\partial x^4}(t_j, \gamma_i)$ and next discretizing the equation (1.16) by implicit-explicit scheme, we have

$$\frac{u_i^{j+1} - u_i^j}{K} = \left[\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}\right] + 6u_i^j\left(1 - u_i^j\right)$$

which yields

$$\Rightarrow u_i^{j+1} = R_u\left[u_{i+1}^j - 2u_i^j + u_{i-1}^j\right] \qquad (1.21)$$
$$+ 6Ku_i^j\left(1 - u_i^j\right)$$
$$+ u_i^j$$

Where the new parameter is defined as $R_u = \frac{K}{h^2}$. To get the numerical solutions, we need to employ the boundary conditions (1.21). The algorithm is developed in FORTRAN 90/95 languages and the version is Plato. In the rest of

the section, the results are presented graphically for further discussion.

The density has been normalized at value taken over the domain at different times. From this graphical structure, we able to see that the solution of $u(t, x)$ is decreasing which means $u(t, x)$ lessens over the domain at time $t = 0.5$. The exact solutions of equation (1.16) using travelling wave scheme which is also represented in Fig. 2 (left) and the error term are visible in Fig. 2 (right). The graph shows that travelling wave solution is also monotonically decreasing while the mesh time step is $\Delta t = K = 0.001$ at time $t = 0.5$ over the domain. It is seen that the solution obtained by implicit-explicit FDM is visually coincides with the exact solution.

Since we have discussed about the nature of the solutions graphically, one additional numerical solutions are presented in Fig. 3 at time $t = 10$ over the domain $\Omega \in (0,1)$.
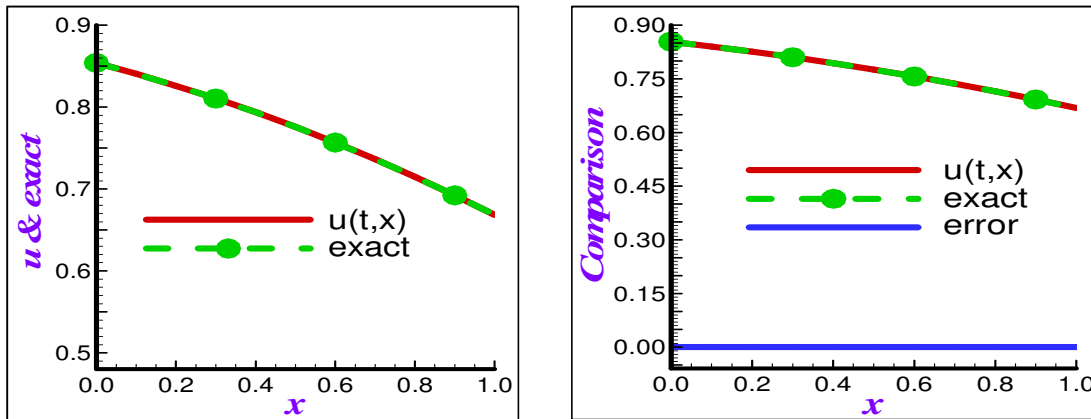

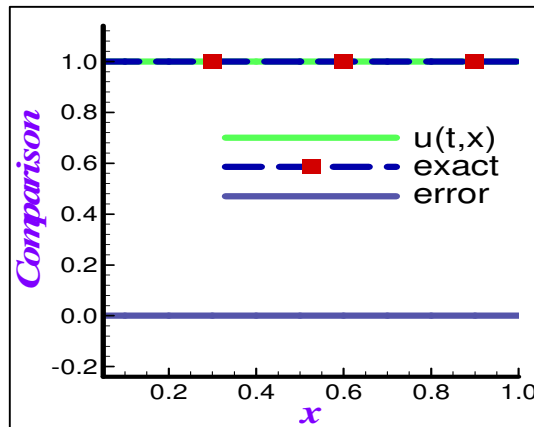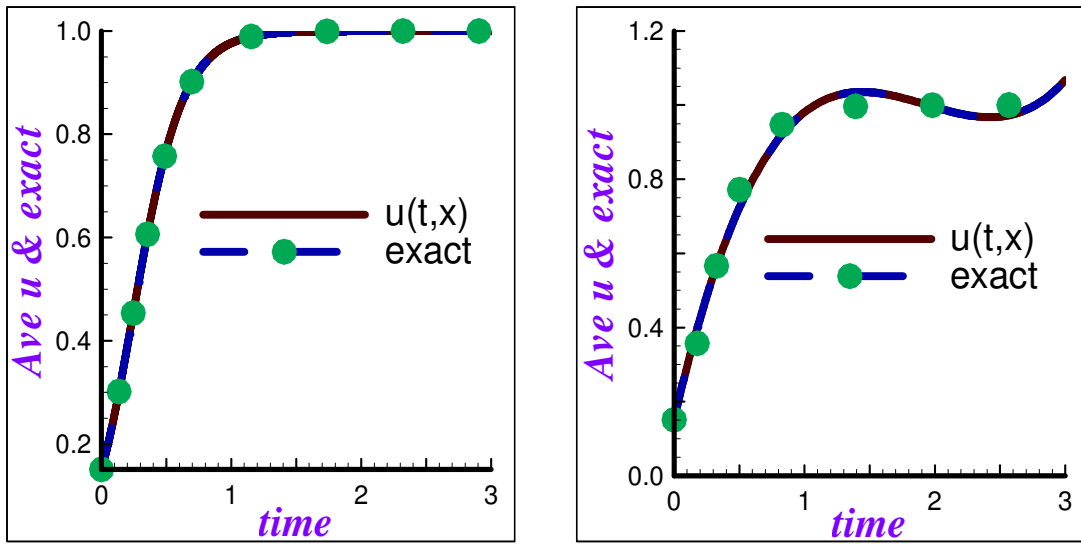**Fig. 2. Comparison of $u(t, x)$ and exact solution with error over the domain at time $t = 0.5$**

Fig. 3. The solution of $u(t, x)$ and exact with error over the domain at time $t = 10.0$
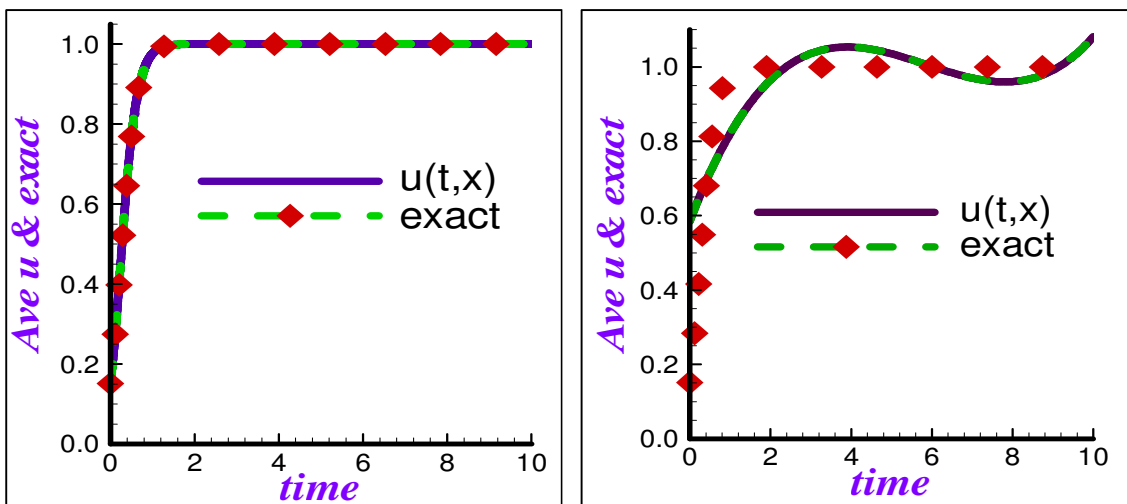
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Fig. 4. Graphical presentation of time vs average solution of $u(t, x)$ in (a) for implicit-explicit FDM and (b) by polynomial-fit and compared with wave solution over Ω at $t = 3.0$

At this stage, we are interested to discuss about average solution produced by implicit-explicit finite-difference method. The solution depicted in Fig. 4 (left) reasonably accurate since there is a better agreement with the exact solution. If we consider the polynomial fit approximations of our available data, the numerical solution is very close to the travelling wave solution over the space at time t = 3.0.

If t varying, we can illustrate the figures as decorated in Fig. 5, where we illustrate average of u(t,x) and compare it with the wave or exact

solution for time t=10.0. The behavior of the solutions is similar to the solutions as shown in Fig. 4. The total illustration of average implicit-explicit solution with wave solution vs time is given in Fig. 6.

Using analytic and numerical solutions, we have obtained the results as depicted in Fig. 6 at $t = 0.5$ and at $t = 3.0$. Time increases from 0.5 to 3.0 and corresponding average of $u(t, x)$ or implicit-explicit finite difference solutions are shown simultaneously in Fig. 6.

Fig. 5. Graphical results of time vs average solution of $u(t, x)$ (a) for implicit-explicit FDM and (b) by polynomial-fit and compared with wave solution over the domain at time $t = 10.0$
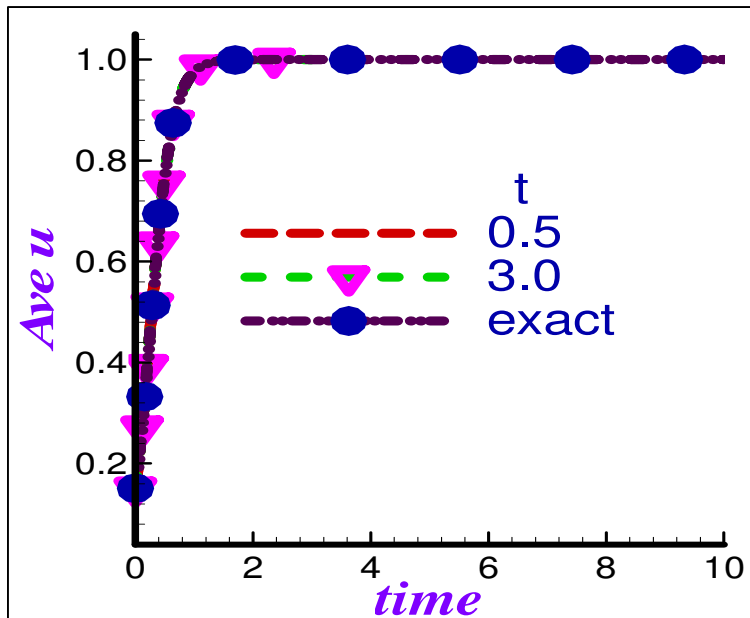
Fig. 6. Average solutions of $u(t,x)$ at time $t = 0.5, t = 3.0$ and compared with exact solution for time $t = 10.0$
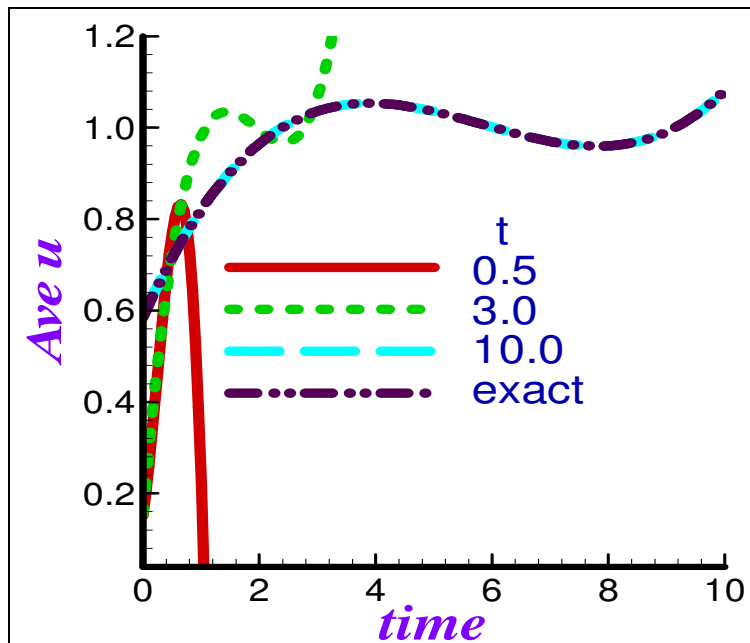

Fig. 7. Multiple plots of $u(t,x)$ at different times and the exact solution for $t = 10.0$

3.4 Stability and Error

Finally, we have displayed multiple plots as shown in Fig. 7, using numerical data for various times in one diagram using polynomial-fit illustrations. While the time is increasing the solutions are closer to the exact solution which is more feasible.

The following sections are concerned about the stability and error test. Implicit-Explicit FDM is valid for $0 < R_u \leq \frac{1}{2}$ only. The readers are referred to check the following Section 4 to know more details about the stability analysis and

convergence of the method. If the solution of the finite difference equations is to be reasonably accurate approximation to the solution of the corresponding nonlinear partial differential equation, then the condition must be satisfied. To satisfy the rest of the part, let us denote E as an exact solution of partial differential equation and the exact solution of finite difference implicit-explicit scheme is denoted by u. Then we consider $e = E - u$, where e is discretization error. Since the simplest implicit-explicit finite difference approximation of equation (1.16) can be written as

$$\frac{u_i^{j+1} - u_i^j}{K} = \left[\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}\right] \qquad (1.22)$$
$$+ 6u_i^j\left(1 - u_i^j\right)$$

The simplification of equation (1.22) is given in equation (1.21). Let us consider now $u_i^{j+1} = E_i^{j+1} - e_i^{j+1}$ and $u_i^j = E_i^j - e_i^j$ at the mesh points. Then putting these in equation (1.22), we obtain

$$e_i^{j+1} = E_i^{j+1} - R_u[E_{i+1}^j - e_{i+1}^j - 2(E_i^j \\ - e_i^j) + E_{i-1}^j - e_{i-1}^j] \qquad (1.23)$$
$$-6K(E_i^j - e_i^j)\left(1 - (E_i^j - e_i^j)\right) - (E_i^j - e_i^j)$$

After using Taylor's theorem [17] in equation (1.23), we can see that $\left|E_i^j - u_i^j\right| \le e_j$ where e_j presents the maximum value of $\left|e_i^j\right|$ which

proves that u converges to E when $R_u \le \frac{1}{2}$ and t is finite.

The implicit-explicit finite difference scheme is unstable when $R_u > \frac{1}{2}$ and conditionally stable if $0 < R_u \le \frac{1}{2}$. Graph of error using approximate and exact solutions is given in Fig. 8.

We show that the errors consisting of difference between exact and approximate solutions using different patterns over the domain at increasing time.

4. DISCUSSION WITH STABILITY TEST AND CONVERGENT ANALYSIS

In the following two sections, we will discuss about the stability of our considered numerical method and the local truncation errors of Implicit-Explicit Finite Difference scheme.

4.1 Stability Analysis and Convergence of the Method

Considering the equation (1.21) which can be represented in tri-diagonal form such as

$$A_i u_{i-1}^j + B_i u_i^j + C_i u_{i+1}^j = D_i \qquad (1.24)$$

Where,

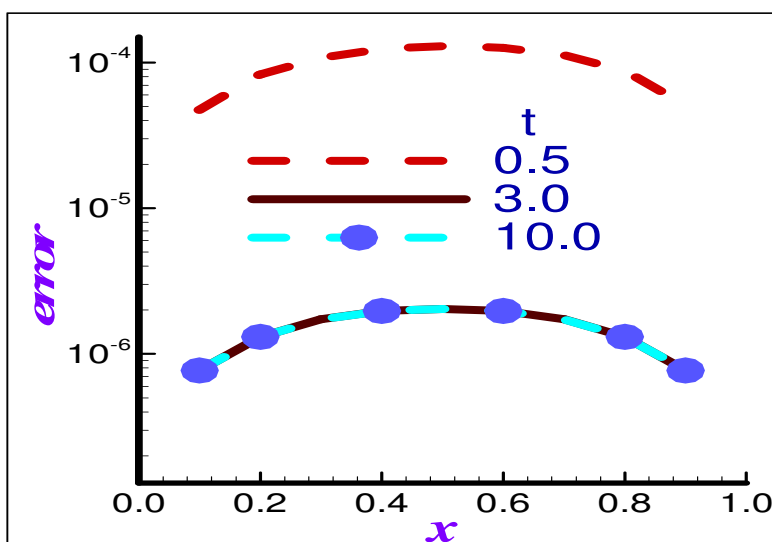$A_i = C_i = R_u$ is a dimensionless constant number and $B_i = 1 - 2R_u + 6K(1 - u_i^j)$.

Fig. 8. Error over the domain at time $t = 0.5, t = 3.0$ and $t = 10.0$

Using boundary conditions with acquiring the whole system of equation, we can construct the matrix form

$$AX = B$$

Where,

$$A = \begin{pmatrix} B_1 & R_u & .... & .... & .... \\ R_u & B_2 & R_u & .... & .... \\ .... & R_u & B_3 & R_u & .... \\ .... & .... & .... & .... & .... \\ .... & .... & .... & .... & .... \\ .... & .... & .... & .... & .... \\ .... & .... & R_u & B_{N-1} & R_u \end{pmatrix}$$ is tri-

diagonal matrix of order

$$m \times n; B = \begin{pmatrix} u_1^j \\ u_2^j \\ \vdots \\ u_N^j \end{pmatrix}$$

withholds all known values and

$$X = \begin{pmatrix} u_1^{j+1} \\ u_2^{j+1} \\ \vdots \\ u_N^{j+1} \end{pmatrix}$$ all unknown values.

Since Fisher's equation is a nonlinear equation and for finding its stability, we have to consider $u_i^j = f$ as a constant such that we can write the tri-diagonal form in terms of constant f. Here the coefficient matrix for (1.21) can be introduced by

$$\begin{pmatrix} Q & R_u & .... & .... & .... \\ R_u & Q & R_u & .... & .... \\ .... & R_u & Q & R_u & .... \\ .... & .... & .... & .... & .... \\ .... & .... & .... & .... & .... \\ .... & .... & .... & .... & .... \\ .... & .... & R_u & B_N & Q \end{pmatrix},$$

Where,

$$Q = 1 - 2R_u + 6K(1 - f)$$ is also a constant.

Since the coefficient matrix is positive definite and also symmetric. One can remark that its eigenvalues are also positive. Here $\lambda_q \leq 1$; $q = 1,2,3 ....$ [29,30]. That is, Eigenvalues of the required matrix must be less than or equal to one for minimizing the errors.

We can also use the fourier method to check if the scheme is stable or not. Assuming a solution of the form

$$u_i^j = a^{(j)}(c)e^{\varphi ich}$$

Where,

c is the wave number, $h = \Delta x$ and $\varphi = \sqrt{-1}$. We introduce here Von Neumann stability condition [31] which is $|L(c)| \leq 1$ for $0 \leq ch \leq \pi$, where $L(c) = \frac{a^{(j+1)}(c)}{a^{(j)}(c)}$; $a(c), L(c)$ are the growth rate of fourier component and an amplification factor respectively. We can say that the explicit scheme is stable if and only if $R_u \leq \frac{1}{2}$ which is also known as conditionally stable. It is also noted that if the implicit and Crank–Nicolson schemes are stable for any values of R_u, then the method is known as unconditionally stable. Substituting the solution of the form u_i^j in (1.24) for implicit-explicit scheme, we get

$$a^{(j+1)}(c)e^{\varphi ich}$$
$$= R_u a^{(j)}(c)e^{\varphi(i+1)ch}$$
$$+ \big[1 - 2R_u$$
$$+ 6K\big(1 - a^{(j)}(c)e^{\varphi ich}\big)\big]a^{(j)}(c)e^{\varphi ich} +$$
$$R_u a^{(j)}(c)e^{\varphi(i-1)ch}$$

$$\Rightarrow L(c) = \frac{a^{(j+1)}(c)}{a^{(j)}(c)}$$
$$= R_u e^{\varphi ch}$$
$$+ \big[1 - 2R_u$$
$$+ 6K\big(1 - a^{(j)}(c)e^{\varphi ich}\big)\big]$$
$$+ R_u e^{-\varphi ch}$$

So by Von Neumann stability condition, $|L(c)| \leq 1$

$$\Leftrightarrow \big|R_u e^{\varphi ch} + \big[1 - 2R_u$$
$$+ 6K\big(1 - a^{(j)}(c)e^{\varphi ich}\big)\big]$$
$$+ R_u e^{-\varphi ch}\big| \leq 1$$
$$\Leftrightarrow |R_u e^{\varphi ch} + [1 - 2R_u] + R_u e^{-\varphi ch}| \leq 1$$
$$\Leftrightarrow |[1 - 2R_u] + 2R_u Cos(ch)| \leq 1$$
$$\Leftrightarrow \left|1 - 2R_u + 2R_u\left(1 - Sin^2\left(\frac{ch}{2}\right)\right)\right| \leq 1$$
$$\Leftrightarrow \left|1 - 2R_u + 2R_u - 4R_u Sin^2\left(\frac{ch}{2}\right)\right| \leq 1$$
$$\Leftrightarrow \left|1 - 4R_u Sin^2\left(\frac{ch}{2}\right)\right| \leq 1$$
$$\Leftrightarrow 0 \leq 4R_u Sin^2\left(\frac{ch}{2}\right) \leq 2$$
$$\Leftrightarrow 0 \leq R_u \leq \frac{1}{2Sin^2\left(\frac{ch}{2}\right)} \text{ for all } 0 \leq ch \leq \pi \text{ and}$$

this is equivalent to $0 \leq R_u \leq \frac{1}{2}$.

4.2 Local Truncation Errors and Consistency

The Implicit-Explicit scheme has local truncation errors. If the step size approaches to zero then

the local truncation error tends to zero. This is the essential condition of the numerical solutions to the continuous solution for better convergence and the method is said to be consistent. For finding the local truncation errors of a numerical method, we can introduce it in a single iteration with the expanding the coefficients by Taylor series method such that

$$u_i^{j+1} = u_i^j + K\frac{\partial u}{\partial t} + \frac{K^2}{2!}\frac{\partial^2 u}{\partial t^2} + \cdots\ldots + O(K)^4;$$

$$u_{i+1}^j = u_i^j + h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} + \cdots\ldots + O(h)^4;$$

$$u_{i-1}^j = u_i^j - h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} - \cdots\ldots + O(h)^4;$$

Where $h = \Delta x$ and $K = \Delta t$. Now substituting all these in the combined Implicit-Explicit method of (1.16), the local truncation errors can be written as

$$lte = \lim_{h,K\to 0}\frac{K^2}{2!}\frac{\partial^2 u}{\partial t^2} + \frac{K^3}{3!}\frac{\partial^3 u}{\partial t^3} + \cdots + \frac{h^4}{4!}\frac{\partial^4 u}{\partial x^4}$$
$$+ \frac{h^6}{6!}\frac{\partial^6 u}{\partial x^6} + \cdots.. = 0;$$

Since the difference between our selected PDE and its FDE representation vanishes as mesh is refined and h, K approaches to 0 for which $\lim_{mesh\to 0} lte = 0$ i.e., local truncation errors become zero, then we conclude that the employed finite difference scheme is consistent.

5. CONCLUSION

It is observed that the density of population diminishes over the domain at certain time and average solutions are coincided for increasing of times. In this paper, we have find that travelling wave solutions exists for $c \geq 2\sqrt{6}$ in the selected Fisher's equation and wave develops with speed $c = 2\sqrt{6}$ in the governing equation. Nonlinear problems like fisher's equations can be solved by Implicit-Explicit schemes. We have generally used Implicit-Explicit method and compared it with travelling wave solutions to justify our solutions. For solving Fisher's equation, this method is simply a powerful technique. The approximate solutions obtained by using this method produce better results as compared with travelling wave solution. However, in neurophysiology, chemical kinetics and population dynamics such types of modeling phenomena like Fisher's equation can be extended for future study.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Fisher RA. The wave of advance of advantageous genes. Annals of Genetics. 1937;7(4):353–369.
2. Kolmogorov A, Petrovskii I, Piscounov N. Study of the diffusion equation with growth of the quantity of matter and its application to a biological problem. Bull. State. Univ. Mos. 1937;1-25.
3. Feng Z. Travelling wave behavior for generalized Fisher equation. Chaos Solitions Fractals. 2008;38:481-488.
4. Feng Z, Li Y. Complex travelling wave solutions to the Fisher equation. Phys. A. 2006;366:115-123.
5. Yuan W, Li Y. The general traveling wave solutions of Fisher equation with degree three. Advances in Mathematical Physics. 2013;5.
6. Tyson JJ, Brazhnik PK. On travelling wave solutions of Fisher's equation in two spatial dimensions. SIAM Journal on Applied Mathematics. 2000;60(2):371-391.
7. Tang S, Weber RO. Numerical study of Fisher's equation by a Petrov-Galerkin finite element method. The Journal of the Australian Mathematical Society. Series B. Applied Mathematics. 1991;33(1):38.
8. Adomian G, Rach R. Solving frontier problems of physics: The decomposition method. Kluwer, J. Math. Anal. Appl. 1986;113(1):199.
9. Canosa J. Diffusion in nonlinear multiplicative media. J. Math. Phys. 1969;10(10):1862-1868.
10. Canosa J. On a nonlinear diffusion equation describing population growth. IBM J. Res. Develop. 1973;17(4):307-313.
11. Hariharan G, Kannan K, Sharma KR. Haar wavelet method for solving Fisher's equation. Appl. Math. Comput. 2009; 211(2):284-292.

12. Ablowitz MJ, Zeppetella. Explicit solution of Fishers equation for a special wave speed. Bull. Math. Biology. 1979;41:835-840.

13. Murray JD. Lectures on nonlinear differential equations model in biology. Oxford University Press, London; 1977.

14. Wazwaz AM, Gorguis A. An analytic study of Fishers equation by using adomian decomposition method. Appl. Math. Comput. 2004;154(3):609-620.

15. Chaturantabut IS, Sorensen DC. Application of POD on dimension reduction of non-linear Fishers equations. 2011;17: 337-353.

16. Smith GD. Numerical solution of partial differential equations: Finite difference methods. Second Edition, Oxford Applied Mathematics and Computing Science Series. 1985;12.

17. Gordon D. Smith. Numerical solution of PDEs. Third Edition, Oxford Applied Mathematics and Computing Science Series. 1985;45.

18. Atta Ullah, Kamal Shah. Numerical analysis of Lane Emden-Fowler equations. Journal of Taibah University for Science. 2018;12(2):180-185.

19. Tahir Khan, et al. Solution of fractional order heat equation via triple Laplace transform in 2 dimensions. Mathematical Methods in the Applied Sciences. 2017;41(2):818–825.

20. Kamal Shah, Hammad Khalil, Rahmat Ali Khan. Analytical solutions of fractional order diffusion equations by natural transform method. Iranian Journal of Science and Technology (Trans Sci: A). 2016;42(3):1479–1490.

21. Amjad Ali, Kamal Shah, Rahmat Ali Khan. Numerical treatment for traveling wave solutions of fractional Whitham-Broer-Kaup equations. Alexandria Engineering Journal. 2018;57(3):1991-1998.

22. Mohamed A. Ramadan, Mohamed R. Ali. New algorithm for solving system of Fredholm integral equations. Electronic Journal of Mathematical Analysis and Applications (EJMAA); 2017.

23. Mohamed A. Ramadan, Adel R. Hadhoud, Mohamed R. Ali. Numerical solutions of singular initial value problems in the second-order ordinary differential equations using Hybrid Orthonormal Bernstein and Block-Pulse Functions. Journal of the Egyptian Mathematical Society. 2018;24(4):45-60.

24. Mohamed A. Ramadan, Mohamed R. Ali. Approximate solutions for fuzzy Volterra integro-differential equations using HOBB method. Journal of Statistics Application and Probability. 2017;4(14):80-94.

25. Mohamed R. Ali. Application of the method of lines for solving the KdV-Burger equation. Journal of Abstract and Computational Mathematics. 2017;2(2):39-51.

26. Mohamed A. Ramadan, Mohamed R. Ali. Application of Bernoulli wavelet method for numerical solution of fuzzy linear Volterra-Fredholm integral equations. Communication in Mathematical Modeling and Applications. 2017;2(3):40-49.

27. Mohamed R. Ali. Approximate solutions for fuzzy Volterra integro-differential equations. Journal of Abstract and Computational Mathematics. 2018;3(2):11-22.

28. Mohamed A. Ramadan, Mohamed R. Ali. A modified gradient-based algorithm for solving extended Sylvester-conjugate matrix equations. Asian Journal of Control. 2018;20(1):228-235.

29. Mitchell AR, Griffiths DF. Handbook of finite difference method in partial differential equations. Wiley-Interscience Publication; 1980.

30. Chandraker V, Jayaraj S. A numerical treatment of Fisher equation. International Conference on Computational Heat and Mass Transfer; 2015.

31. Davis M. Finite difference methods, Part II. Numerical solution of PDEs.