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Lie's Theorems on Soluble Leibniz Algebras

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Abstract

New proofs for Lie's theorems on soluble Leibniz algebras.

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1 Introduction

Several results on Lie algebras have known extensions to the case of Leibniz algebras. See for example [1],[2],[3],[4],[5], ... Lie's theorems have been extended to Leibniz algebra, however most of these results follow the pattern that from a Leibniz algebra L we get a Lie algebra obtained by the quotient $L/\mathit{Ess}(L)$ where $\mathit{Ess}(L)$ is generated by the square of the elements of L, also called "partie essentielle" [6].

The purpose of this paper is to adapt the proof of Lie's theorem on soluble Lie algebra [7], and to give proof which also covers in a more general way all soluble Leibniz algebras.

Here, our approach is based on the work of J. E. Hymphreys [7], we find the main results, having evaded the difficulty that $\mathit{Ess}(L) \neq \{0\}$ for a non Lie Leibniz algebra. Section 2 is devoted to reminders on definitions and general results. In Section 3 some results on Leibniz modules are given. Proofs of Lie' theorems are then generalized in Section 4.

2 Preliminaries

Throughout this paper, F will be an algebraically closed field of characteristic zero. All vector spaces and algebras will be finite dimensional over F. The dimension of an F-vector space V will be denoted $\dim_F V$. Note the sum of two vector subspaces V_1, V_2 by $V_1 \dot{+} V_2$ and direct sum by $V_1 \oplus V_2$.

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Definition 2.1. (Leibniz algebra) [6]

A Leibniz algebra is a vector space L equipped with a bilinear map $[-,-] : L\times L \longrightarrow L$, satisfying the Leibniz identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y] \text{ for any } x,y,z \in L. \tag{2.1}$$

If the condition $[x,x] = 0$ is fulfilled, the Leibniz identity is equivalent to the so-called Jacobi identity. Therefore Lie algebras are particular cases of Leibniz algebras. A morphism of Leibniz algebras is a linear map $\phi : L_1 \longrightarrow L_2$ such that for any $x,y \in L_1$ $\phi([x,y]) = [\phi(x),\phi(y)]$.

It follows from the Leibniz identity that in any Leibniz algebra one has

$$[y,[x,x]] = 0,\ [z,[x,y]] + [z,[y,x]] = 0, \text{ for all } x,y,z \in L.$$

Definition 2.2. (Ideal) A subspace H of a Leibniz algebra L is called left (respectively right) ideal if for $a \in H$ and $x \in L$ one has $[x,a] \in H$ (respectively $[a,x] \in H$). If H is both left and right ideal, then H is called (two-sided) ideal.

If V is a vector space, let $End_F(V)$ denotes the set of all endomorphisms of V. An action of L on $End_F(V)$ is a linear map of L on $End_F(V)$.

Definition 2.3. (Representation) Let L be a Leibniz algebra and V a vector space. V is an L-module if there are:

- a left action, $l : L \longrightarrow End_F(V),\ x \mapsto l_x$
- a right action, $r : L \longrightarrow End_F(V),\ x \mapsto r_x$,
  such that:

$$\begin{array}{rcl} r_{[x,y]} &=& r_y r_x - r_x r_y, \\ l_{[x,y]} &=& r_y l_x - l_x r_y, \\ l_{[x,y]} &=& r_y l_x + l_x l_y, \\ 0 &=& l_x l_y + l_x r_y. \end{array}$$

For x in L, $r_x(v)$ will be denoted by vx and $l_x(v)$ will be denoted by xv. The triplet (l,r,V) is called a representation of L on V. Now if L is a Leibniz algebra, we have the adjoint representation "(Ad, ad, L)" defined as follows: for all x and y in L, $ad_x : L \longrightarrow L$, $y \longmapsto [y,x]$ and $Ad_x : L \longrightarrow L$, $y \longmapsto [x,y]$

Remark 2.1. For $x \in L$, $ad_x : L \longrightarrow L$ is a derivation of L i.e. for all x,y,z in L,

$$ad_x([y,z]) = [ad_x(y),z] + [y, ad_x(z)].$$

For $x \in L$, $Ad_x : L \longrightarrow L$ is an anti-derivation of L i.e. for all x,y,z in L,

$$Ad_x([y,z]) = [Ad_x(y),z] - [Ad_x(z),y].$$

If L is a Lie algebra, for all x,y in L,

$$Ad_x(y) = -ad_x(y).$$

For an arbitrary algebra and for all non negative integer n let us define the sequences:

(i) $D^1(L) = L^{[1]} = L^2$, $D^{n+1}(L) = L^{[n+1]} = [L^{[n]}, L^{[n]}]$;

(ii) $L^1 = L$, $L^{n+1} = [L^1, L^n] + [L^2, L^{n-1}] + \cdots + [L^{n-1}, L^2] + [L^n, L^1]$.

Definition 2.4. [1]

An algebra L is called solvable if there exists $m \in \mathbb{N}^*$ such that $D^m(L) = L^{[m]} = \{0\}$.
An algebra L is called nilpotent if there exists $s \in \mathbb{N}^*$ such that $L^s = \{0\}$.

Definition 2.5. Let X be a subset of V. The subspace of V spanned by the subset X will be denoted by *Span*(X) and note *Span*$(\{v\})$ by *Span*(v).

The following two lemmas can be found in [8] but for self contained we give proofs here.

Lemma 2.1. *Let L be a Leibniz algebra and (l, r, V) a representation of L. We have:*

i) $\text{Ess}(V) = \text{Span}\,(\{xv + vx, \text{ for all } (x, v) \in L \times V\})$ *is a submodule of the L-module V and* $\text{Ess}(V) \subseteq \ker(l_x)$ *for all $x \in L$;*

ii) *if L is viewed as a L-module equipped with the two actions Ad and ad, then* $\text{Ess}(L)$ *is an ideal of L and $Ad_z(v), ad_z(v)$ are elements of* $\text{Ess}(V)$ *for all $(z, v) \in \text{Ess}(L) \times V$.*

Proof. Clearly *Ess*(V) is a subspace of V, *Ess*(L) is a subspace of L.

i) Let $l_x(v) + r_x(v)$ be a generator of *Ess*(V). For z in L we have:

$$\begin{aligned} l_z\left(l_x(v) + r_x(v)\right) &= l_z\left(l_x(v)\right) + l_z\left(r_x(v)\right) \\ &= l_{[z,x]}(v) - r_x\left(l_z(v)\right) - l_{[z,x]}(v) + r_x\left(l_z(v)\right) = 0. \end{aligned}$$

$$\begin{aligned} r_z\left(l_x(v) + r_x(v)\right) &= r_z\left(l_x(v)\right) + r_z\left(r_x(v)\right) \\ &= l_{[x,z]}(v) + l_x\left(r_z(v)\right) + r_{[x,z]}(v) + r_x\left(r_z(v)\right) \\ &= \left(l_{[x,z]}(v) + r_{[x,z]}(v)\right) + \left(l_x\left(r_z(v)\right) + r_x\left(r_z(v)\right)\right). \end{aligned}$$

It follows that $l_z\left(l_x(v) + r_x(v)\right) = 0$ and $r_z\left(l_x(v) + r_x(v)\right)$ is a sum of generators of *Ess*(V) so *Ess*(V) is stable under the actions of l_z, r_z for all $z \in L$. Then *Ess*(V) is a submodule.

ii) Applying the first result to the L-module L equipped with the two actions Ad and ad; we have that *Ess*(L) is an ideal of L and for all generator $z = [a, b] + [b, a]$ of *Ess*(L) and all v in V:

$$\begin{aligned} l_z(v) &= l_{[a,b]}(v) + l_{[b,a]}(v) \\ &= r_b\left(l_a(v)\right) + l_a\left(l_b(v)\right) + r_a\left(l_b(v)\right) + l_b\left(l_a(v)\right) \\ &= \left(l_a\left(l_b(v)\right) + r_a\left(l_b(v)\right)\right) + \left(l_b\left(l_a(v)\right) + r_b\left(l_a(v)\right)\right). \end{aligned}$$

So that $Ad_z(v)$ is a sum of two generators of *Ess*(M).

$$\begin{aligned} r_z(v) &= r_{[a,b]}(v) + r_{[b,a]}(v) \\ &= r_b\left(r_a(v)\right) - r_a\left(r_b(v)\right) + r_a\left(r_b(v)\right) - r_b\left(r_a(v)\right) = 0. \end{aligned}$$

It follows that $ad_z(v)$ and $Ad_z(v)$ belong to *Ess*(V).

□

Lemma 2.2. *Let L be a Leibniz algebra. We have:*

i) *for any derivation D of L, $D(\text{Ess}(L)) \subseteq \text{Ess}(L)$;*

ii) *for any anti-derivation $\widetilde{D}$ of L, $\widetilde{D}(\text{Ess}(L)) = \{0\}$.*

Proof. For any generator $[x, y] + [y, x]$ of *Ess*(L) we have:

i) $D([x, y] + [y, x]) = D([x, y]) + D([y, x])$
$\qquad = ([D(x), y] + [y, D(x)]) + ([D(y), x] + [x, D(y)])$.
So $D([x, y] + [y, x])$ is a sum of two generators of *Ess*(L) and hence $D($*Ess*$(L)) \subseteq$ *Ess*(L).

ii) $\widetilde{D}([x, y] + [y, x]) = \widetilde{D}([x, y]) + \widetilde{D}([y, x])$
$\qquad = [\widetilde{D}(y), x] - [\widetilde{D}(x), y] + [\widetilde{D}(x), y] - [\widetilde{D}(y), x]$
$\qquad = 0$.

The lemma is proved. □

Definition 2.6. Let f be in $End_F V$ and W a subsapce of V, say the subspace W is f-stable if $f(W) \subseteq W$.

For example, let (l, r, V) be a representation of L. For any $x \in L$, the submodule *Ess*(V) is l_x-stable and r_x-stable.

Here we recall some results on linear algebra.

Lemma 2.3. *Let $\alpha \in F$, f an element of $End_F(V)$ such that* Ess(V) *is f-stable and u_1 a vector of* Ess$(V) \setminus \{0\}$. *Set*

$$U = \text{Span}(u_1 \cdots, u_m \cdots) \text{ and } W = \text{Span}(w_1, \cdots, w_m, \cdots);$$

where for any integer $i \geq 1$

$$u_{i+1} = f(u_i), \text{ and } w_i = u_{i+1} - \alpha u_i.$$

Then $W \subseteq U$ are subspaces of V, f-stable and for a suitable α in F we have $\dim_F W = \dim_F U - 1$.

Proof. Let m be the least integer such that $(u_1, \cdots, u_{m+1})$ is linearly dependant and hence we have $u_{m+1} = \sum_{j=1}^{m} \alpha_j u_j$ for α_j in F.

We have also:

$$\begin{aligned}
u_{m+1} - \alpha u_m &= (\alpha_m - \alpha)\, u_m + \sum_{j=1}^{m-1} \alpha_j u_j \\
&= (\alpha_m - \alpha)\,(u_m - \alpha u_{m-1}) + \left(\alpha_m \alpha - \alpha^2\right) u_{m-1} + \sum_{j=1}^{m-1} \alpha_j u_j \\
&= (\alpha_m - \alpha)\,(u_m - \alpha u_{m-1}) + \\
&\qquad + \left(\alpha_{m-1} + \alpha_m \alpha - \alpha^2\right) u_{m-1} + \sum_{j=1}^{m-2} \alpha_j u_j \\
&= (\alpha_m - \alpha)\,(u_m - \alpha u_{m-1}) \\
&\qquad + \left(\alpha_{m-1} + \alpha_m \alpha - \alpha^2\right)(u_{m-1} - \alpha u_{m-2}) \\
&\qquad + \left(\alpha_{m-2} + \alpha_{m-1}\alpha + \alpha_m \alpha^2 - \alpha^3\right) u_{m-2} + \sum_{j=1}^{m-3} \alpha_j u_j.
\end{aligned}$$

Step by step we obtain that:

$$w_m = u_{m+1} - \alpha u_m = \sum_{j=2}^{m-1} P_j(\alpha)\,(u_j - \alpha u_{j-1}) + P_1(\alpha) u_1$$

where $P_j(t) = \sum_{k=j}^{m} \alpha_k t^{k-j} - t^{m-j+1}$ for $j = 1, \cdots, m$.

Let α be a root of the polynom $P_1(t)$, then $w_m = \sum_{j=2}^{m} P_j(\alpha) w_{j-1}$.

$B' = (w_1, \cdots, w_{m-1})$ is a basis of W.

We can note that V and W are f-stable. ☐

Lemma 2.4. *Let $\alpha \in F$, f an endomorphism of V such that* Ess(V) *is f-stable and u_1 a vector of $V \setminus$* Ess(V). *Set*

$$U' = \text{Ess}(V) \dot{+} \text{Span}(u_1 \cdots, u_m \cdots); W' = \text{Ess}(V) \dot{+} \text{Span}(w_1, \cdots, w_m, \cdots)$$

where for any integer $i \geq 1$

$$u_{i+1} = f(u_i), w_i = u_{i+1} - \alpha u_i.$$

Then $W' \subseteq U'$ are subspaces of V, f-stable and for a suitable α in F we have $\dim_F W' = \dim_F U' - 1.$

Proof. Proof is similar to the proof of the Lemma 2.3. □

Remark 2.2. Let $N = dim_F V$ be the dimension of vector space V. Let f an endomorphism of V and $u_1 \neq 0$.

If $u_1 \in$ *Ess*(V), put $W^0 = V$. Throughout Lemma 2.3, we can constructed an f-stable subspace W^1 which is of dimension $N-1$. By repeating the process with the vector space W^1, it follows that W^2 is an f-stable subspace which dimension is $N-2$.
And so on, a decreasing chain of f-stable subspaces are constructed:

$$W^0 = V \supsetneq W^1 \supsetneq \cdots \supsetneq W^{N-2} \supsetneq W^{N-1} \supsetneq \{0\}.$$

It is consequently clear that any generator w of W^{N-1} satisfies $f(w) = \lambda w$ for some $\lambda \in F$.

If $u_1 \in V \setminus$ *Ess*(V), put $W^0 = V$. Throughout Lemma 2.4, we can constructed an f-stable subspace W^1 which is of dimension $N-1$. By repeating the process with the vector space W^1, it follows that W^2 is an f-stable subspace which dimension is $N-2$.
And so on, a decreasing chain of f-stable subspaces are constructed:

$$W^0 = V \supsetneq W^1 \supsetneq \cdots \supsetneq W^{N-2} \supsetneq W^{N-1} \supsetneq \{0\}.$$

It is also clear that any generator w of W^{N-1} satisfies $f(w) = \lambda w$ for some $\lambda \in F$.

3 On Leibniz Modules

Definition 3.1. (Complement of subspace)
Let V be an L-module and v a vector in V. Let *Comp*(v) denotes the complement in $Fv \dot{+}$ *Ess*(V) of the one dimensional subspace Fv.

Thus if $\{0\} \subsetneq Fv \subsetneq Fv \dot{+}$ *Ess*(V) we can obtain a basis $\{v, v_1, \cdots, v_p\}$ of $Fv \dot{+}$ *Ess*(V) such that *Comp*$(v) =$ *Span*$(\{v_1, \cdots, v_p\})$ and if *Ess*$(V) \subseteq Fv$ we set *Comp*$(v) = \{0\}$.

Definition 3.2. (Dual space and pseudo-weights)
Let L be any vector space. Then we denote the set of F-linear maps from L to F by L^* and call it the dual space of L. Let K be an ideal of a Leibniz algebra L and V a finite-dimensional L-module. For $\lambda \in L^*$ and set

$$V_{K,l,\lambda} = \{v \in V,\ (l_k - \lambda(k)1_V)(v) \in \text{Comp}(v) \text{ for all } k \in K\};$$
$$V_{K,r,\lambda} = \{v \in V,\ (r_k - \lambda(k)1_V)(v) \in \text{Comp}(v) \text{ for all } k \in K\};$$
$$V_{l,\lambda} = \{v \in V,\ (l_x - \lambda(x)1_V)(v) \in \text{Comp}(v) \text{ for all } x \in L\};$$
$$V_{r,\lambda} = \{v \in V,\ (r_x - \lambda(x)1_V)(v) \in \text{Comp}(v) \text{ for all } x \in L\}.$$

A pseudo-weight of L (on V) is an element $\lambda \in L^*$ such that $V_{l,\lambda} \dot{+} V_{r,\lambda} \neq \{0\}$.

When $V_{l,\lambda} \neq \{0\}$ (respectively $V_{r,\lambda} \neq \{0\}$), we call it a pseudo-weight spaces.

Lemma 3.1. *Let V be an L-module, $(\lambda, \mu) \in L^* \times L^*$ and K an ideal of L. Then*

i) Ess$(V) \subseteq V_{K,l,0}$;

ii) $V_{K,l,\lambda} \cap V_{K,l,\mu} \neq \{0\}$ *if and only if* $\lambda = \mu$.

Proof. **i)** Let $0 \neq w_1$ a vector of *Ess*(V); w_1 is an eigenvector of l_x with eigenvalue equals 0. (see the Lemma 2.1).

ii) Let $v \neq 0$ a vector in $V_{K,l,\lambda} \cap V_{K,l,\mu}$, then for any $k \in K$ there are $w_2(k), w_3(k)$ in *Comp*(v) such that:

$$\begin{cases} kv = \lambda(k)v + w_2(k); \\ kv = \mu(k)v + w_3(k). \end{cases}$$

So $0 = (\lambda - \mu)(k)\, v + (w_2(k) - w_3(k)) \in Kv \oplus$ *Comp*(v) which implies $w_2(k) = w_3(k)$ and $\lambda(k) = \mu(k)$ for any $k \in K$.

□

Remark 3.1. Let $\lambda \in L^*$ and $\lambda \neq 0$. Let $v_0 \in V_{K,l,\lambda}$. (So $v_0 \notin$ *Ess*(V)). Let $0 \neq w \in$ *Ess*(V). We have, for all $k \in K$, $l_k(v_0) = \lambda(k)v_0 + w_1(k)$ where $w_1(k) \in$ *Comp*$(v_0) =$ *Ess*(V).
Set $v_1 = 2v_0 + w$, for all $k \in K$, we have $l_k(v_1) = \lambda(k)(v_1) + (2w_1(k) + \lambda(k)w)$. Clearly *Comp*$(v_1) =$ *Ess*(V) and $v_1 \in V_{K,l,\lambda}$.
But we have $w = v_1 - 2v_0 \in$ *Ess*$(V) \subset V_{l,0}$ and thus $v_1 - 2v_0 \notin V_{K,l,\lambda}$.
$V_{K,l,\lambda}$ is not a vector subspace of a vector space V if $\lambda \neq 0$.
Let v_0, v_1 be two vectors of the pseudo-weight space $V_{K,l,\lambda}$ (with $\lambda \neq 0$). Let $\alpha_0 \in F$, $\alpha_1 \in F$ such that $\alpha_0 v_0 + \alpha_1 v_1 \notin$ *Ess*(V) then $\alpha_0 v_0 + \alpha_1 v_1 \in V_{K,l,\lambda}$. So *Ess*$(V) \dot{+} V_{K,l,\lambda}$ is a vector subspace of V.

Before introduce the following lemma, we shall note that: for all $x \in L$, basic results on linear algebra imply that there is a basis $B_0 = (e_0, \cdots, e_p)$ of *Ess*(V) and a $(p+1)$-uplet $(\lambda_0, \cdots, \lambda_p)$ of F^{p+1}such that,

$$\begin{aligned} &r_x(e_0) = \lambda_0 e_0; \\ &r_x(e_i) - \lambda_i e_i \in \textit{Span}(e_0, \cdots, e_{i-1}),\ i = 1, \cdots, p; \\ &l_x(e_i) = 0,\ i = 0, \cdots, p. \end{aligned} \tag{3.1}$$

Let M_{0x} be the matrix of the restriction of l_x to *Ess*(V) and N_{0x} the matrix of the restriction of r_x to *Ess*(V), relative to basis B_0, we have:

$$M_{0x} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = 0,\ N_{0k} = \begin{pmatrix} \lambda_0 & a_{0,1} & a_{0,2} & \cdots & a_{0,p} \\ 0 & \lambda_1 & a_{1,2} & \cdots & a_{1,p} \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \lambda_{p-1} & a_{p-1,p} \\ 0 & 0 & \cdots & 0 & \lambda_p \end{pmatrix}.$$

Lemma 3.2. *If* $\lambda \neq 0$ *then* $V_{K,l,\lambda} \subseteq V_{K,r,-\lambda} \subseteq \mathrm{Ess}(V) \dot{+} V_{K,l,\lambda}$.

Proof. Since $\lambda \neq 0$, there are some $k \in K$ and $v \neq 0$ in $V_{K,l,\lambda}$ such that $\lambda(k) \neq 0$ and $l_k(v) = kv = \lambda(k)v + w(k)$ for some $w(k)$ in *Comp*(v).

$\lambda(k) \neq 0$ implies that $v \notin$ *Ess*(V) and *Comp*$(v) =$ *Ess*(V).
Also we have $w_1(k) = l_k(v) + r_k(v) \in$ *Ess*(V) which implies that $r_k(v) = -\lambda(k)v + w_1(k) - w(k)$. Indeed $w_1(k) - w(k) \in$ *Ess*$(V) =$ *Comp*(v), so $v \in V_{K,r,-\lambda}$.

Now let $v \neq 0$ in $V_{K,r,-\lambda}$ such that $r_k(v) = vk = -\lambda(k)v + w(k)$ for some $w(k)$ in *Comp*(v).

If $v \in$ *Ess*(V) we have $v \in V_{K,l,\lambda} \dot{+}$ *Ess*(V), else $v \notin$ *Ess*(V) and then *Comp*$(v) =$ *Ess*(V), in such case we have $l_k(v) = \lambda(k)v + w_1(k) - w(k)$ and indeed $w_1(k) - w(k) \in$ *Ess*$(V) =$ *Comp*(v). Then we have $v \in V_{K,l,\lambda}$ and so $v \in$ *Ess*$(V) \dot{+} V_{K,l,\lambda}$.

□

Remark 3.2. If *Ess*$(V) = \{0\}$, V is a Lie-module, $V_{K,l,\lambda}$ is the the set of eigenvectors (of l_k associated with the eigenvalue $\lambda(k)$) as defined in [7, page 16].

$$W = V_{K,l,\lambda} = V_{K,r,-\lambda} = \{v \in V,\ l_k(v) = -r_k(v) = \lambda_k v \text{ for all } k \in K\}$$

Example 3.3. *Let $L = \mathbb{C}x + Cy$ be the two dimensional complex Leibniz algebra which generators satisfy $[x,x] = [y,y] = [y,x] = 0; [x,y] = x$. And let L viewed as an L-module.*

Let $\lambda \in L^*$ define by $\lambda(x) = 0; \lambda(y) = 1$. Then:

$$V_{ad,\lambda} = \mathbb{C}x, \qquad V_{ad,0} = \mathbb{C}y, \qquad V_{Ad,0} = L, \qquad V_{Ad,-\lambda} = \{0\}.$$

We note that $V_{ad,\lambda} \neq V_{Ad,-\lambda}$ and $V_{ad,0} \neq V_{Ad,0}$.

Proposition 3.1. *Let $\lambda \neq 0$ in L^*, $v \neq 0$ in $V_{K,l,\lambda}$, x be in L and $\text{Ess}(V) = \text{Span}(e_p, \cdots, e_1)$, the sequence defined by $u_1 = v$, $u_{n+1} = xu_n = x^n v,\ n \in \mathbb{N}^*$ satisfies for all $k \in K$, the equations:*

$$\begin{aligned} ku_1 &= \lambda(k)u_1 + \underbrace{w^1(k)}_{\in \text{Span}(e_p,\cdots,e_1)}; \\ ku_n &= \lambda(k)u_n + \underbrace{w^n(k)}_{\in \text{Span}(u_{n-1},\cdots,u_1,e_p,\cdots,e_1)} \qquad (\text{for } n \geq 2). \end{aligned}$$

and $U = \text{Span}(e_1, \cdots, e_p, u_1, \cdots, u_n, \cdots)$ is invariant under K and under x.

Proof. It is obvious that the subspace defined by $U = \textit{Span}(e_1, \cdots, e_p, u_1, \cdots, u_n, \cdots)$ is invariant under x. Since, $\lambda \neq 0$, $V_{K,l,\lambda} \cap V_{K,l,0} = \{0\}$ so $v \notin \textit{Ess}(V)$ and the subspace $\textit{Comp}(v) = \textit{Comp}(u_1) = \textit{Ess}(V) = \textit{Span}(e_p, \cdots, e_1)$.

For all $k \in K$, $ku_1 - \lambda(k)u_1 \in \textit{Comp}(u_1)$ and then we have, for all $k \in K$, $ku_1 = \lambda(k)u_1 + \underbrace{w^1(k)}_{\in \textit{Ess}(V)}.$

Let $u_2 = xu_1$ then,

$$\begin{aligned} ku_2 &= k(xu_1) \\ &= [k,x]\,u_1 - (ku_1)x \\ &= \lambda([k,x])u_1 + w^1([k,x]) - (\lambda(k)u_1 + w^1(k))x \\ &= \lambda(k)xu_1 + \underbrace{\lambda([k,x])u_1 + w^1([k,x]) - w^1(k)x - \lambda(k)(u_1x + xu_1)}_{\in \textit{Span}(u_1,e_p,\cdots,e_1)} \\ &= \lambda(k)u_2 + \underbrace{\lambda([k,x])u_1 + w^1([k,x]) - w^1(k)x - \lambda(k)(u_1x + xu_1)}_{\in \textit{Span}(u_1,e_p,\cdots,e_1)}. \end{aligned}$$

We have for all $k \in K$, $ku_2 = \lambda(k)\,(u_1) + \underbrace{w^1(k)}_{\in \textit{Span}(u_1,e_p,\cdots,e_1)}.$

Suppose, by induction that for all $k \in K$ and $n > 1$,

$$ku_n = \lambda(k)u_n + \underbrace{w^n(k)}_{\in \textit{Span}(u_{n-1},\cdots,u_1,e_p,\cdots,e_1)}.$$

Then,

$$\begin{aligned} ku_{n+1} &= k(xu_n) \\ &= [k,x]u_n - (ku_n)x \\ &= \lambda([k,x])u_n + w^n([k,x]) - (\lambda(k)u_n + w^n(k))x \\ &= \lambda(k)xu_n + \lambda([k,x])u_n - \lambda(k)(u_nx + xu_n) + \underbrace{w^n([k,x]) - w^n(k)x}_{\in \textit{Span}(u_{n-1},\cdots,u_1,e_p,\cdots,e_1)} \\ &= \lambda(k)u_{n+1} + \underbrace{\lambda([k,x])u_n + w^n([k,x]) - w^n(k)x - \lambda(k)(u_nx + xu_n)}_{\in \textit{Span}(u_n,\cdots,u_1,e_p,\cdots,e_1)}, \end{aligned}$$

thus for all $k \in K$, $ku_{n+1} = \lambda(k)u_{n+1} + \underbrace{w^{n+1}(k)}_{\in \textit{Span}(u_n,\cdots,u_1,e_p,\cdots,e_1)}.$

Induction is done.

Induction shows also that $U'_n = \textit{Span}(u_n, \cdots, u_1, e_p, \cdots, e_1)$ is invariant under K.

□

Let $v \in V_{K,l,\lambda} \setminus$ *Ess*(V) and $u_1 = v$.

Let m be the least integer, such that $(u_1, \cdots, u_{m+1})$ is linearly dependent.
$U =$ *Span*$(u_m, \cdots, u_1, e_p, \cdots, e_1)$ is invariant under K and under the action of x. So U is an L-module. According to Equation 3.1 and Proposition 3.1, we have that the matrix M_k of l'_x (the restriction of l_k to U), for any $k \in K$, with respect to the basis $B = (e_1, \cdots, e_p, u_1, \cdots, u_m)$ is an upper triangular matrix with all diagonal entries being nul or equal to $\lambda(k)$:

$$M_k = \left(\begin{array}{ccc|cccc} 0 & \cdots & 0 & a_{1,p+1} & * & \cdots & a_{1,p+m} \\ \vdots & \ddots & \vdots & \vdots & & & \vdots \\ 0 & \cdots & 0 & a_{p,p+1} & \cdots & * & a_{p,p+m} \\ \hline 0 & \cdots & 0 & \lambda(k) & a_{p+1,p+2} & \cdots & a_{p+1,p+m} \\ 0 & & \vdots & 0 & \lambda(k) & \ddots & \vdots \\ \vdots & & 0 & \vdots & \ddots & \ddots & a_{p+m-1,p+m} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \lambda(k) \end{array}\right)$$

Let N_k be the matrix of r'_x (the restriction of r_k to U), with respect to the basis B.
Due to $l_k(v) + r_k(v) \in$ *Ess*(V), we have

$$N_k = \left(\begin{array}{cccc|cccc} \lambda_0 & b_{1,2} & \cdots & b_{1,p} & b_{1,p+1} & * & \cdots & b_{1,p+m} \\ 0 & \lambda_1 & \ddots & \vdots & & & & \\ \vdots & \ddots & \ddots & b_{p-1,p} & \vdots & & & \vdots \\ 0 & \cdots & 0 & \lambda_p & b_{p,p+1} & \cdots & * & b_{p,p+m} \\ \hline 0 & \cdots & & 0 & -\lambda(k) & a_{p+1,p+2} & \cdots & a_{p+1,p+m} \\ 0 & & & \vdots & 0 & -\lambda(k) & \ddots & \vdots \\ \vdots & & & 0 & \vdots & \ddots & \ddots & a_{p+m-1,p+m} \\ 0 & \cdots & & 0 & 0 & \cdots & 0 & -\lambda(k) \end{array}\right)$$

Now, we can prove the following

Lemma 3.4. $\lambda([k,x]) = 0$ *for all* $k \in K$ *and all* $x \in L$.

Proof. If $\lambda(k) = 0$ for all $k \in K$, there is nothing to prove.

Else there is an $k' \in K$, $v \in V_{K,l,\lambda}$ such that $\lambda(k') \neq 0$ and $(l_{k'} - \lambda(k')1_V)(v)$ lies in *Ess*(V).

Then $u_1 = l_{k'}(v) = k'v = \lambda(k')v + w(k') \notin$ *Ess*(V). Let U be spanned by the linearly independant family $B = (e_1, \cdots, e_p, u_1, \cdots, u_m)$. U is invariant under K and under x, so it is invariant under the whole Leibniz subalgebra $K \dot{+}$*Span*(x) of L. For every element k in K, the commutator $[k,x]$ is contained in K, so the matrix $M_{[k,x]}$ of its action on U with respect to the basis B is upper triangular with $\lambda([k,x])$ or zero on the diagonal. On the other hand, since $l_{[k,x]} = r_x l_k - l_k r_x$, its matrix is the commutator of the matrix N_x and M_k, so in particular its trace is zero. Thus $tr(M_{[k,x]}) = m\lambda([k,x]) = 0$ implies $\lambda([k,x]) = 0$ and we have proved the Lemma.

Note that we have proved at the same time that

$$U = \textit{Span}(u_1, \cdots, u_m) \subseteq V_{K,l,\lambda}.$$

□

Lemma 3.5. *The subspace* Ess$(V) \dot{+} V_{K,l,\lambda}$ *is a submodule of* V.

Proof. For v in *Ess*$(V) \dot{+} V_{K,l,\lambda}$, we have to show that $l_x(v)$ and $r_x(v)$ belong to *Ess*$(V) \dot{+} V_{K,l,\lambda}$. Thanks to Lemma 3.1, *Ess*(V) is a submodule lying in the kernel of l_x for all x in L; so we will deal with a vector u not in *Ess*(V).

if $\lambda \equiv 0$, *Ess*$(V)\dot{+}V_{K,l,0} = V_{K,l,0}$ and let $u \in V_{K,l,0}$ then, we have $k\,(xu) = [k,x]u - (ku)x = w'([k,x]) - w'(k)x$ where $w'([k,x]), w'(k)$ belong to the subspace *Comp*$(u) =$ *Ess*$\,(V)$.
This shows that $k^2\,(xu) = k\,(w'([k,x]) - w'(k)x) = 0$ and then xv lies in $V_{K,l,0}$. Moreover the relation $l_x(v) + r_x\,(x) \in$ *Ess*$\,(V)$ tells us that also $r_x(v)$ lies in $V_{K,l,0}$.
So that $V_{K,l,0}$ is a submodule.

if $\lambda \not\equiv 0$, let $u \in V_{K,l,\lambda}$ then;

$$\begin{aligned} Al &= k(xu) - \lambda(k)(xu) = [k,x]u - (ku)x - \lambda(k)(xu) \\ &= \lambda([k,x])u + w([k,x]) - \lambda(k)(ux + xu) - w(k)x. \end{aligned}$$

Let us note that $w([k,x])$ and $w(k)$ ly in *Ess*(V), then

$$w([k,x]) - \lambda(k)(ux + xu) - w(k)x \in \textit{Ess}(V).$$

By the Lemma 3.4, $\lambda([k,x]) = 0$, which implies that $(l_k - \lambda(k)1_K)\,(xu)$ lies in *Ess*(V) and so $xu = l_x(u) \in$ *Ess*$(V)\dot{+}V_{K,l,\lambda}$. Indeed, since $l_k(u) + r_k(u) \in$ *Ess*(V), also $ux = r_x(u) \in$ *Ess*$(V)\dot{+}V_{K,l,\lambda}$.

□

Let $u_1 \in$ *Ess*$(V) \setminus \{0\}$ and x an element of L not in $[L,L]$.
Consider the sequence $u_1, u_2 = r_x(u_1), \cdots, u_{n+1} = r_x(u_n) = r_x^n\,(u_1)\,, \cdots$ for all $n \in \mathbb{N}^*$.

Proposition 3.2. *Let L be a solvable Leibniz F-algebra, V an L-module and $\mathcal{K}$ be an ideal of L of codimension one such that $L = \mathcal{K} \oplus Fx$ for $x \notin [L,L]$. Suppose there are a non-zero vector $u_1 \in$ Ess(V) and the functions $\varrho, \zeta : \mathcal{K} \longrightarrow F$ such that $l_h(u_1) = \zeta(h)u_1 = 0_V$, $r_h(u_1) = \varrho(h)u_1$ for all $h \in \mathcal{K}$. Then $U =$ Span$(u_1, \cdots, u_n, \cdots)$ is an L-module in which lies a common eingevector i. e. a vector $v \in U$ along with the functions $\varrho, \varsigma : L \longrightarrow F$ such that $l_y(v) = \varsigma(y)v = 0_V$, $r_y(v) = \varrho(y)v$ for all $y \in L$.*

Proof. Note that by induction we have for all $i \geq 1$ and all $h \in \mathcal{K}$:

$$\begin{aligned} r_h(u_{i+1}) &= r_h(r_x(u_i)) = r_x(r_h(u_i)) + r_{[h,x]}(u_i)) \\ &= \varrho(h)r_x(u_i) + \varrho([h,x])u_i = \varrho(h)u_{i+1}. \end{aligned}$$

Note also that r_x is an endomorphism of the vector space L and u_1 satisfies the hypotheses of Lemma 2.3. Thanks to Lemma 2.3 and Remark 2.2 there is a vector $w \in$ *Span*$(u_1, \cdots, u_m, \cdots)$ which satisfies $r_x(w) = \lambda w$. Make ϱ an element of L^* by defined $\varrho(h + \alpha x) = \varrho(h) + \alpha\lambda$ for all $h \in \mathcal{K}$, $\alpha \in F$. Clearly we have, for any $y \in L$, $r_y(w) = \varrho(y)w$. Since $w \in$ *Ess*(L), for any $y \in L$, $l_y(w) = 0$.
Hence Fw is one dimensional submodule and w a common eingevector. □

Let $u_1 \notin$ *Ess*(V) and x an element of L not in $[L,L]$.
Let $B_0 = (e_1, \cdots, e_p)$ be a basis of *Ess*(V).
Consider the sequence $u_1, u_2 = r_x(u_1), \cdots, u_{n+1} = r_x(u_n) = r_x^n\,(u_1)\,, \cdots$ for all $n \in \mathbb{N}^*$.

Proposition 3.3. *Let L be a solvable Leibniz F-algebra, V an L-module and $\mathcal{K}$ be an ideal of L of codimension one such that $L = \mathcal{K} \oplus Fx$ for $x \notin [L,L]$. Suppose there are a non-zero vector $u_1 \notin$ Ess(V) and the functions $\varrho, \zeta : \mathcal{K} \longrightarrow F$ such that $l_h(u_1) = \zeta(h)u_1 = -\varrho(h)u_1$, $r_h(u_1) = \varrho(h)u_1$ for all $h \in \mathcal{K}$. Then $U' =$ Span$(e_1, \cdots, e_p, u_1, \cdots, u_m, \cdots)$ is an L-module in which lies a common eingevector i. e. a vector $v \in U$ and the functions $\varrho, \varsigma : L \longrightarrow F$ such that $l_y(v) = -\varrho(y)v$, $r_y(v) = \varrho(y)v$ for all $y \in L$.*

Proof. Note that by induction we have for all $i \geq 1$ and all $h \in \mathcal{K}$:

$$\begin{aligned} r_h(u_{i+1}) &= r_h(r_x(u_i)) = r_x(r_h(u_i)) + r_{[h,x]}(u_i)) \\ &= \varrho(h)r_x(u_i) + \varrho([h,x])u_i = \varrho(h)u_{i+1}. \end{aligned}$$

Note also that r_x is an endomorphism of the vector space L and u_1 satisfies the hypotheses of Lemma 2.4. Thanks to Lemma 2.4 and Remark 2.2 there is a vector $w \in$ *Span*$(u_1, \cdots, u_m, \cdots)$ which satisfies $r_x(w) = \lambda w$. Make ϱ an element of L^* by defined $\varrho(h + \alpha x) = \varrho(h) + \alpha\lambda$ for all $h \in \mathcal{K}$, $\alpha \in F$. Clearly we have, for any $y \in L$, $r_y(w) = \varrho(y)w$. If $w \in$ *Ess*(L), then for any $y \in L$, $l_y(w) = 0$ else $l_y(w) = -r_y(w) = -\varrho(y)w$. Hence Fw is one dimensional submodule and w a common eingevector. □

4 On Lie's Theorems

Theorem 4.1. *[Lie.] Let L be a solvable Leibniz algebra over an algebraically closed field F of characteristic zero, and V an L-module. Then we can find a basis B of V such that for every x in L the matrix of l_x and r_x, with respect to the base B, are upper triangular matrix.*

By induction on $dim_F L + dim_F V$ this reduces to the following :

Theorem 4.2. *[Lie.] Under the same hypotheses, there exists a common eigenvector v and the functions $\chi, \varsigma : L \longrightarrow F$ such that*

$$l_y(v) = \chi(y)v,\ r_y(v) = \varsigma(y)v \text{ for all } y \in L.$$

Proof. We will procced by induction on $\dim_F L + \dim_F V$. Let suppose that $L \neq \{0\}$ and $V \neq \{0\}$.

Let $\dim_F L + \dim_F V = 2$, then $\dim_F L = \dim_F V = 1$, then $L = Fx$ and $V = Fv$ and results are obvious.

Let us suppose by induction that the result holds for any couple (L, V), where L is solvable Leibniz algebra and V an L-module, with $\dim_F L + \dim_F V \leq n$.

Consider now a couple (L, V) with $\dim_F L + \dim_F V = n + 1$. Since L is solvable, pick an element x of $L \setminus [L, L]$. Denote by $\mathcal{K}$ the complement in L of the one dimensional subspace Fx. We have $L = Fx \oplus \mathcal{K}$.

Notice also that $[L, L] \subseteq \mathcal{K}$ and $\mathcal{K}$ is an ideal of L of codimension one.

We have $\dim_F \mathcal{K} + \dim_F V = n$, so by induction there is a non-zero vector $u_1 \in V$ and the functions $\varrho, \zeta : \mathcal{K} \longrightarrow F$ such that $l_h(u_1) = \zeta(h)u_1$, $r_h(u_1) = \varrho(h)u_1$ for all $h \in \mathcal{K}$.

If $u_1 \in$ *Ess*(V) then by Proposition 3.2 there is a common eingevector v along with the functions $\varrho, \varsigma : L \longrightarrow F$ such that $l_y(v) = \varsigma(y)v = 0_V$, $r_y(v) = \varrho(y)v$ for all $y \in L$.

Else if $u_1 \notin$ *Ess*(V) then by Proposition 3.3 there is a common eingevector v along with the functions $\varrho, \varsigma : L \longrightarrow F$ such that $l_y(v) = \varsigma(y)v$, $r_y(v) = \varrho(y)v$ for all $y \in L$.

So proofs of theorems are done. □

Remark 4.1. If $v \in$ *Ess*(V) then $\chi(y) = 0$ for all $y \in L$
else $\chi(y) = -\varsigma(y)$ for all $y \in L$ since $l_y(v) + r_y(v) = (\chi(y) + \varsigma(y))v \in$ *Ess*(V).

In the case of the adjoint representation ($l = Ad, r = ad, V = L$), a flag of subspaces stable under L is a chain of ideals. This proves the following corollary.

Corollary 4.3. *If L is a solvable Leibniz algebra there exists a chain of ideals $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$ such that $dim L_i = i$.*

Corollary 4.4. *If L is solvable Leibniz algebra, then $x \in [L, L]$ implies that x is ad-nilpotent. In particular $[L, L]$ is nilpotent.*

Proof. Find a flag of ideals as in the Corollary 4.3.

Relative to a basis$(x_1, \cdots, x_n)$ of L where $(x_1, \cdots, x_i)$ spans L_i, the matrix of ad_x is an upper triangular matrix.

Thus the matrix of $ad_{[x,y]} = [ad_x, ad_y]$ is a strictly upper triangular matrix. Hence ad_x is nilpotent for $x \in [L, L]$. The last statement follows by Engel's theorem for Leibniz algebras [2]. □

5 Conclusion

We give here a panoramic exposure on solvable Leibniz algebras. By this paper, we bring another way, more elegant and simple, to give the proof of Lie' theorems on solvable Leibniz algebras, which generalizes that known on Lie algebras.

Competing Interests

The authors declare that no competing interests exist.

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