

# First Integral Method: A General Formula for Nonlinear Fractional Klein-Gordon Equation Using Advanced Computing Language

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## Abstract

In this article, a general formula of the first integral method has been extended to celebrate the exact solution of nonlinear time-space differential equations of fractional orders. The proposed method is easy, direct and concise as compared with other existent methods.

## Keywords

**First Integral Method, Exact Solution, Fractional Klein-Gordon Equation**

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## 1. Introduction

Partial differential equations arise frequently in the formulation of fundamental laws of nature and in the mathematical analysis of a wide variety of problems in applied mathematics, mathematical physics, and engineering science. This subject plays a central role in modern mathematical sciences. In recent years, it has turned out that many phenomena in fluid mechanics, viscoelasticity, biology, physics, engineering and other areas of science can be successfully modeled by the use of fractional derivatives and integrals [1] [2]. In this article, we study fractional differential equations associated with derivatives. Such a kind of equation appears in many problems. In particular, we have found fractional differential equations related to the classical Klein-Gordon equation [3].

## 2. Preliminaries and Basic Definitions and First Integral Method

The Jumarie's modified Riemann-Lionville derivative, of order  $\alpha$ , can be defined by the following expression

$$D_s^\alpha f(s) = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} [f(\xi) - f(0)] d\xi, \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} [f(\xi) - f(0)] d\xi, 0 < \alpha < 1 \\ (f^n(x))^{\alpha-n}, n \leq \alpha < n+1, n \geq 1 \end{cases} \quad (1)$$

Moreover, some properties for the modified Riemann-Liouville derivative can be given as follows:

$$D_s^\alpha s^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} s^{r-\alpha}, \quad (2)$$

$$D_s^\alpha (f(s)g(s)) = f(s)D_s^\alpha g(s) + g(s)D_s^\alpha f(s), \quad (3)$$

$$D_s^\alpha (f[g(s)]) = f'_g[g(s)]D_s^\alpha g(s) = D_s^\alpha f[g(s)](g'(t))^\alpha \quad (4)$$

Consider the time fractional differential equation with independent variables  $X = (x_1, x_2, \dots, x_m, t)$  and a dependent variable  $u$ ,

$$W(u, D_t^\alpha u, u_{x_1}, u_{x_2}, u_{x_3}, D_t^{2\alpha} u, u_{x_1 x_1}, u_{x_2 x_2}, u_{x_3 x_3}, \dots) = 0 \quad (5)$$

Using the variable transformation

$$u(x_1, x_2, \dots, x_m, t) = u(\xi), \xi = x_1 + l_1 x_2 + \dots + l_{m-1} x_m + \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} \quad (6)$$

where  $k, l_i$  and  $\lambda$  are constants to be determined later. The fractional differential Equation (5) is reduced to a nonlinear ordinary differential equation

$$F(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \dots) = 0 \quad (7)$$

We assume that Equation (7) has a solution in the form

$$u(\xi) = x(\xi) \quad (8)$$

and introduce a new independent variable  $y(\xi) = x'(\xi)$ , which leads to a new system of ordinary differential equations

$$\begin{aligned} x'(\zeta) &= y(\zeta) \\ y'(\zeta) &= F(x(\zeta), y(\zeta)) \end{aligned} \quad (9)$$

By using the division theorem for two variables in complex domain  $C[x, y]$  which is based on the Hilbert-Nullstellensatz theorem [4], we can obtain a first integral to Equation (9) which can applied to Equation (7) to obtain a first-order ordinary differential equation. An exact solution to Equation (5) is then obtained by solving the equations that are obtained from applying Equation (9) into Equation (7). Now, we wish to quickly recall the division theory.

**Theorem 2.1 [The Division Theorem]**

Suppose that  $P(x, y)$  and  $Q(x, y)$  are polynomials of two variables  $x$  and  $y$  in  $C[x, y]$  and  $P(x, y)$  is irreducible in  $C[x, y]$ . If  $Q(x, y)$  vanishes at all points of  $P(x, y)$ , then there exists a polynomial  $G(X, Y)$  in  $C[x, y]$  such that  $Q(X, Y) = P(X, Y)G(X, Y)$ .

### 3. The First Integral Method: A General Formula

We discuss the problem by using the first integral method and consider the general formula [5]:

$$u''(\xi) - su' - k - lu(\xi) - mu^2(\xi) - nu^3(\xi) - fu^4(\xi) - ru^5(\xi) = 0 \quad (10)$$

where  $s, k, l, m, n, f, r$  are real constant.

Substituting Equation (7) in Equation (6), we get the system

$$\begin{aligned} X_\xi(\xi) &= Y(\xi), \\ Y_\xi(\xi) &= sY + k + lX(\xi) + mX^2(\xi) + nX^3(\xi) + fX^4(\xi) + rX^5(\xi) \end{aligned} \tag{11}$$

Now, we apply the division theorem. Suppose that  $X(\zeta)$  and  $Y(\zeta)$  are nontrivial solution of equation (11). Then

$$q(x, y) = \sum_{i=0}^M a_i(x) y^i = 0 \tag{12}$$

which is an irreducible Polynomial in the complex domain  $C[X, Y]$ , thus

$$q[X(\zeta), Y(\zeta)] = \sum_{i=0}^M a_i(X(\zeta)) Y^i(\zeta) = 0 \tag{13}$$

$a_i(X) (i = 0, 1, 2, \dots, M)$  are polynomial and  $a_M(X) \neq 0$ , Equation (13) is called the first integral method. There exists a polynomial  $g(X)X + h(X)Y$  in the complex domain  $C[x, y]$  such that

$$\frac{dq}{d\zeta} = \frac{dq}{dX} \cdot \frac{dX}{d\zeta} + \frac{dq}{dY} \cdot \frac{dY}{d\zeta} = (g(X)X + h(X)Y) \sum_{i=0}^M a_i(X) Y^i \tag{14}$$

which can be written as

$$\begin{aligned} &\sum_{i=0}^M a_i^*(X) Y^{i+1} + \sum_{i=0}^M i a_i(X) Y^{i-1} (sY + k + lX + mX^2 + nX^3 + fX^4 + rX^5) \\ &= (g(X) + h(X)Y) \sum_{i=0}^M a_i(X) Y^i \end{aligned} \tag{15}$$

by comparing the coefficients of  $Y^i (i = M + 1, M, \dots, 1, 0)$  on both sides of Equation (15), we obtain

$$a_M^* = h(X) a_M(X) \tag{16a}$$

$$a_{M-1}^* = g(X) a_M(X) + h(X) a_{M-1}(X) \tag{16b}$$

⋮

$$a_0^*(X) = -2a_2(X)(sY + k + lX + mX^2 + nX^3 + fX^4 + rX^5) = g(X) a_0(X) \tag{16x}$$

$$a_1(X)(sY + k + lX + mX^2 + nX^3 + fX^4 + rX^5) = g(X) a_0(X) \tag{16y}$$

from (12a), we deduce that  $a_M(X)$  is a constant and  $h(X) = 0$ , we take  $a_M(X) = 1$ , and by balancing the degrees of  $g(X), a_1(X)$ , and  $a_0(X)$ , we find the degree of  $g(X)$ . Now we consider the following cases.

**Case (1)**

When  $M = 1$ , in Equation (12), then the Equation (13) becomes

$$a_1^* = h(X) a_M(X) \tag{17a}$$

$$a_0^* + sa_1(X) = g(X) a_1(X) + h(X) a_0(X) \tag{17b}$$

$$a_1(X)(k + lX + mX^2 + nX^3 + fX^4 + rX^5) = g(X) a_0(X) \tag{17c}$$

Since  $a_i(X) (i = 0, 1)$  are polynomial, then from Equation (14a) we deduce that  $a_1(X)$  is constant and  $h(X) = 0$

For simplicity, we take  $a_1(X) = 1$ . By balancing the degrees of  $g(X)$  and  $a_0(X)$ , we conclude that  $\deg(g(X)) = 2$

Suppose that  $g(X) = A_2X^2 + A_1X + A_0$ , then we can find  $a_0(X)$ .

$$a_0(X) = B_0 + (A_0 - s)X + \frac{A_1 X^2}{2} + \frac{A_2 X^3}{3} \tag{18}$$

where  $B_0$  is an arbitrary integration constant. By substituting  $a_0(X)$ ,  $a_1(X)$ , and  $g(X)$  in Equation (17c), and setting all the coefficients of powers of  $X$  to zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$r = \frac{1}{3}A_2^2, f = \frac{5}{6}A_1A_2, n = \frac{4}{3}A_2A_0 + \frac{A_1^2}{2} - sA_2, m = \frac{3}{2}A_0A_1 - sA_1, l = A_0^2 - sA_0, k = 0, B_0 = 0 \tag{19}$$

Substituting Equation (16) in Equation (10), we obtain

$$Y = \frac{-2A_2X^3 - 3A_1X^2 - 6(A_0 - s)X}{6} \tag{20}$$

By combining Equation (20) with Equation (11), we find the exact solution of Equation (11).

**Case (2)**

When  $M = 2$ , in Equation (15), then the Equation (16) became

$$a_2^*(X) = h(X)a_2(X) \tag{21a}$$

$$a_1^*(X) + 2a_2(X)s = g(X)a_2(X) + h(X)a_1(X) \tag{21b}$$

$$a_0^*(X) + a_1(X)s + 2a_2(k + lX + mX^2 + nX^3 + fX^4 + rX^5) = g(X)a_1(X) + h(X)a_0(X) \tag{21c}$$

$$a_1(X)(k + lX + mX^2 + nX^3 + fX^4 + rX^5) = g(X)a_0(X) \tag{21d}$$

Since  $a_2(X)$  ( $i = 0, 1, 2$ ) are polynomial, then from Equation (14a) we deduce that  $a_2(X)$  is constant and  $h(X) = 0$

For simplicity, we take  $a_2(X) = 1$ . By balancing the degrees of  $g(X)$ ,  $a_1(X)$  and  $a_0(X)$ , we conclude that  $\deg(g(X)) = 2$ ,

Suppose that  $g(X) = A_2X^2 + A_1X + A_0$ , then we find  $a_1(X)$  and  $a_0(X)$

$$a_1(X) = B_0 + (A_0 - 2s)X + \frac{A_1 X^2}{2} + \frac{A_2 X^3}{3} \tag{22}$$

$$\begin{aligned} a_0(X) = & B_1 + [A_0B_0 - 2k]X + \frac{1}{2}[A_1B_0 + A_0^2 - 3A_0s + 2s^2 - 2l]X^2 \\ & + \frac{1}{3}[A_2B_0 + \frac{3}{2}A_0A_1 - \frac{5A_1s}{2} - 2m]X^3 + \frac{1}{4}[\frac{4}{3}A_0A_2 + 2A_2s + \frac{A_1^2}{2} - 2n]X^4 \\ & + \frac{1}{5}[\frac{5}{6}A_1A_2 - 2f]X^5 + \frac{1}{6}[\frac{A_2^2}{3} - 2r]X^6 \end{aligned} \tag{23}$$

where  $B_0, B_1$  are arbitrary integration constants.

By substituting  $a_0(X) \cdot a_1(X)$  and  $g(X)$  in (21d), and setting all the coefficients of powers of  $X$  to zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$k = 0, l = \frac{A_0^2}{4} - \frac{sA_0}{2}, m = \frac{9A_0A_1}{16} - sA_1, n = \frac{4A_0A_2 + A_1^2}{8} + \frac{7sA_2}{5}, f = \frac{5A_1A_2}{16}, r = \frac{A_2^2}{12}, B_1 = 0, B_0 = 0 \tag{24}$$

Using Equation (23) in Equation (12), we obtain two equal roots for  $Y$ .

Note that

$$a_0(X) = \frac{a_1^2(X)}{4}, \text{ then } Y = \frac{-2A_2X^3 - 3A_1X^2 - 6(A_0 - s)X}{12}, \tag{25}$$

Combining Equation (24) with Equation (11), we find the exact solution of Equation (11).

**Case 3**

When  $M = 3$ , in Equation (15), then the Equation (16) become

$$a_3^*(X) = h(X)a_3(X) \tag{26a}$$

$$a_2^*(X) + 3sa_3(X) = g(X)a_3(X) + h(X)a_2(X) \tag{26b}$$

$$a_1^*(X) + 2sa_2(X) + 3a_3(k + lX + mX^2 + nX^3 + fX^4 + rX^5) = g(X)a_2(X) + h(X)a_1(X) \tag{26c}$$

$$a_0^*(X) + sa_1(X) + 2a_2(k + lX + mX^2 + nX^3 + fX^4 + rX^5) = g(X)a_1(X) + h(X)a_0(X) \tag{26d}$$

$$a_1(X)(k + lX + mX^2 + nX^3 + fX^4 + rX^5) = g(X)a_0(X) \tag{26e}$$

Since  $a_2(X) (i = 0, 1, 2)$  are polynomial, then from Equation (17a) we deduce that  $a_2(X)$  is constant and  $h(X) = 0$

For simplicity, we take  $a_2(X) = 1$ . Balancing the degrees of  $g(X), a_1(X)$  and  $a_0(X)$ . We conclude that  $\deg(g(X)) = 2$

Suppose that  $g(X) = A_2X^2 + A_1X + A_0$ , then we find  $a_2(X), a_1(X)$  and  $a_0(X)$ :

$$a_2(X) = B_0 + (A_0 - 3s)X + \frac{A_1X^2}{2} + \frac{A_2X^3}{3} \tag{27}$$

$$a_1 = \frac{1}{6} \left[ -3r + \frac{1}{3}A_1^2 \right] X^6 + \frac{1}{5} \left[ -3f + \frac{5}{6}A_1A_2 \right] X^5 + \frac{1}{4} \left[ -3n + \frac{1}{2}A_1^2 + \frac{4}{3}A_0A_2 + \frac{sA_2}{3} \right] X^4 + \frac{1}{3} \left[ -3m + \frac{3}{2}A_0A_1 - (s+3)A_1 \right] X^3 + \frac{1}{2} \left[ -3l + A_0^2 + 6s^2 - 5sA_0 \right] X^2 \tag{28}$$

$$a_0 = B_1 + \frac{1}{9} \left[ \frac{A_2^3}{18} - \frac{1}{6}A_2r \right] X^9 + \frac{1}{8} \left[ \frac{2}{9}A_1A_2^2 - \frac{4}{15}fA_2 \right] X^8 + \frac{1}{7} \left[ \frac{7}{18}A_0A_2^2 + \frac{7}{24}A_1^2A_2 - \frac{5}{12}nA_2 - \frac{1}{10}fA_1 + \frac{1}{2}rA_0 \right] X^7 + \frac{1}{6} \left[ \frac{1}{2}A_1A_2^2 - \frac{2}{3}mA_2 + \frac{1}{2}A_0A_1A_2 + \frac{1}{8}A_1^3 - \frac{1}{4}nA_1 + \frac{4}{5}A_0f \right] X^6 + \frac{1}{5} \left[ \frac{5}{6}A_0^2A_2 + \frac{1}{2}A_2^2A_2 + \frac{1}{8}A_0A_1^2 - \frac{7}{6}lA_2 - \frac{1}{2}mA_1 + \frac{1}{4}nA_0 + \frac{25sA_0A_2}{6} + \frac{41s^2A_2}{12} + (9n - 4A_1)s \right] X^5 + \frac{1}{4} \left[ -2lA_1 - \frac{13A_0A_1}{3} + 2ms + \frac{4sA_1}{3} - \frac{lA_2}{3} + \frac{A_0^2A_1}{2} \right] X^4 + \frac{1}{3} \left[ -\frac{5}{2}lA_0 + \frac{9ls}{2} - 3sA_0^2 + \frac{11s^2A_0^2}{2} + \frac{A_0^2}{2} - 3s^2 \right] X^3 \tag{29}$$

where  $B_0, B_1$  are arbitrary integration constants.

Substituting  $a_0(X) \cdot a_1(X)$  and  $g(X)$  in Equation (21d), and setting all the coefficients of powers of  $X$  to zero, we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$k = 0, l = \frac{A_0^2}{4} + 3A_0s - 4s^2, m = \frac{9A_0A_1}{16} + \frac{8sA_1}{3}, n = \frac{4A_0A_2 + A_1^2}{12} + \frac{A_2s}{3}, f = \frac{5A_1A_2}{16}, r = \frac{A_2^2}{12}, B_1 = 0, B_0 = 0 \tag{30}$$

Substituting Equation (23) in Equation (12), we obtain three equal roots for  $Y$

Note that

$$a_1(X) = \frac{a_2^2(X)}{3}, a_0(X) = \frac{a_2^3(X)}{27},$$

$$\text{then } Y = \frac{-2A_2X^3 - 3A_1X^2 - 6(A_0 - s)X}{18}, \tag{31}$$

Combining Equation (24) with Equation (11), we find the exact solution of Equation (11).

By the integrating Equations (20), (25) and (30), we can find different solutions of Equation (10), and the exact solution of the general formula in Equation (10) are given by combining Equations (20), (24), (30), with (11) and integrating respect with respect to  $\xi$ .

We can apply Theorem 3.1 to studying some time fractional differential equations.

### 4. Applications

#### The Space-Time Fractional Klein-Gordon Equation [6]:

Consider the nonlinear fractional Klein-Gordon equation,

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} + au(x,t) + cu^3(x,t), t > 0, 0 < \alpha \leq 1$$

$$u(x,0) = \sqrt{\frac{a}{c}} \tan \left[ \sqrt{\frac{a}{2(\lambda^2 - l^2)}} lx \right] \tag{32}$$

For our purpose, we introduce the following transformations

$$u(x, y, t) = u(\xi), \xi = lx + \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}, \tag{33}$$

where  $l, \lambda$  are constant.

Substituting Equation (33) in Equation (32)

$$u'' + \frac{a}{l^2 - \lambda^2} u + \frac{c}{l^2 - \lambda^2} u^3 = 0, \tag{34}$$

which is the same as in the form of Equation (10) where

$$s = 0, k = 0, l = \frac{a}{\lambda^2 - l^2}, m = 0, n = \frac{c}{\lambda^2 - l^2}, f = 0, r = 0 \tag{35}$$

$$A_0 = \pm \frac{a}{\sqrt{2c(\lambda^2 - l^2)}}, A_1 = 0, A_2 = \pm \sqrt{\frac{2c}{\lambda^2 - l^2}} \tag{36}$$

$$y = \frac{2A_0 + A_1x^2}{2} \tag{37}$$

Integrating with respect to  $\xi$  then

$$u_1(x,t) = \sqrt{\frac{b}{A}} \tan \left( \sqrt{Ab} \left( lx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + \xi_0 \right) \right), A > 0, b > 0 \tag{38}$$

$$u_2(x,t) = \sqrt{\frac{b}{A}} \tanh \left( \sqrt{Ab} \left( lx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + \xi_0 \right) \right), A > 0, b < 0 \tag{39}$$

$$u_3(x,t) = \sqrt{\frac{b}{A}} \tan \left( \sqrt{Ab} \left( \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} - lx + \xi_0 \right) \right), A < 0, b > 0 \tag{40}$$

$$u_4(x, t) = \sqrt{\frac{b}{A}} \tanh \left( \sqrt{Ab} \left( \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} - lx + \xi_0 \right) \right), A < 0, b < 0 \quad (41)$$

where  $A = \pm \frac{a}{\sqrt{2c(\lambda^2 - l^2)}}, b = \pm \sqrt{\frac{c}{2(\lambda^2 - l^2)}}.$

where  $\xi_0$  is an arbitrary constant.

Consider the initial condition (32), we can see that the generalized the nonlinear fractional Klein-Gordon Equation (32) have the exact solution

$$u(x, t) = u_1(x, t) = \sqrt{\frac{b}{A}} \tan \left( \sqrt{Ab} \left( lx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} + \xi_0 \right) \right)$$

Comparing our results with that of results [6], shows the novelty of our solution. See **Appendix A Figure 1**.

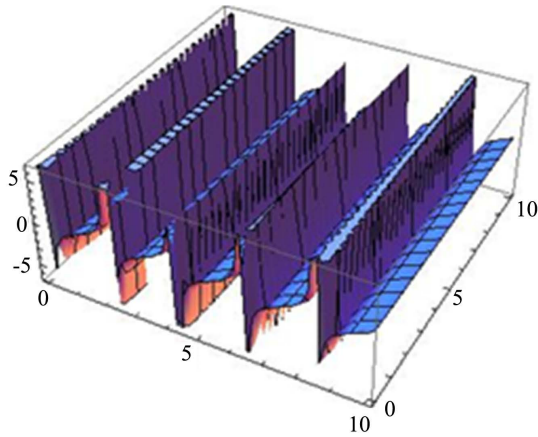
## 5. Conclusion

A general formula of the first integral method is used successfully to solve systems of nonlinear fractional differential equations. The performance of this method is reliable, effective and gives more solutions. We have extended the general formula to solve other fractional differential equations.

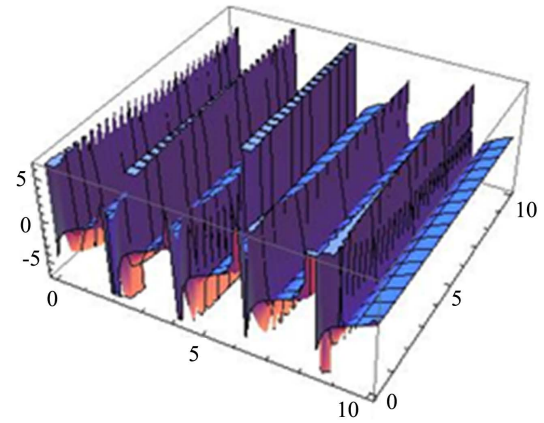
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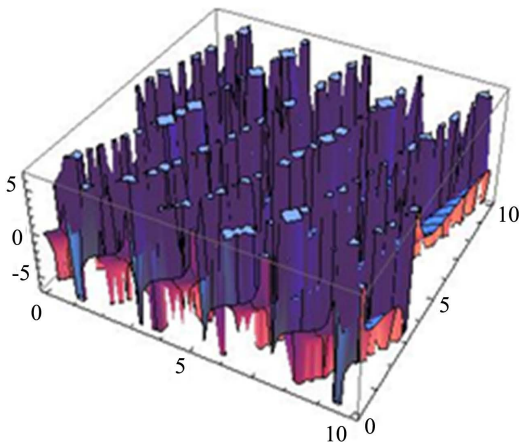
### Appendix A Figure1



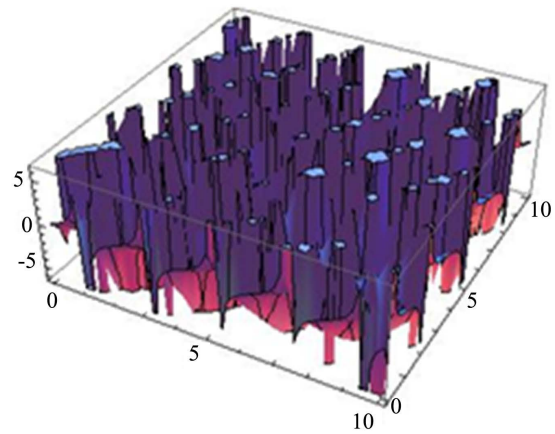
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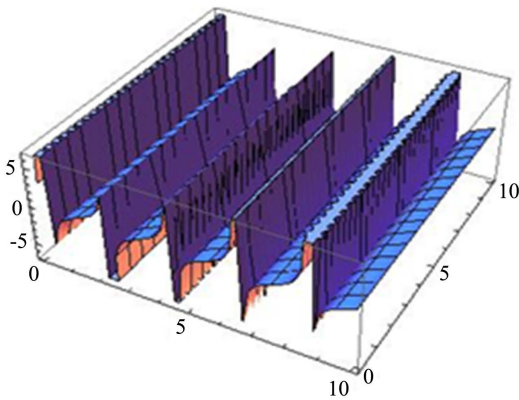
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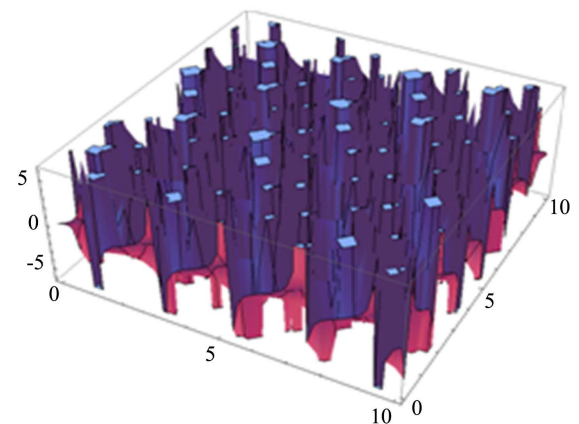
$\alpha = 0.5, \lambda = 1.9, L = 1.5, b = A = 1$



$\alpha = 0.75, \lambda = 1.9, L = 1.5, b = A = 1$



$\alpha = 0.0, \lambda = 1.9, L = 1.5, b = A = 1$



$\alpha = 1.0, \lambda = 1.9, L = 1.5, b = A = 1$

**Figure 1.** Graph of the solution  $u(x, t)$  corresponding to the values of  $\alpha, \lambda, L$  and  $b$  as shown in the caption.