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The Action of a Left Amenable Semi-topological Semigroup on a Weakly Compact Subset of a Banach Space with Normal Structure

Foad Naderi[1*]

[1]*Department of Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran 14115-134, Iran.*

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Abstract

Let S be left amenable semi-topological semigroup such that its weak almost periodic compactification S^w is a topological semigroup. We show that every separately continuous, non-expansive and equicontinuous action of S on a weakly compact convex subset of a Banach space with normal structure has a fixed point.

Keywords: Non-expansive mapping; semi-topological semigroup; amenable; left reversible; weakly almost periodic function.

1 Introduction

Let K be a subset of a Banach space E. A self mapping T on K is said to be *non-expansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in K$.

If H and K are non-empty subsets of a Banach space E and H is bounded, for $k \in K$, define $r(H,k) = sup\{ \|h - k\| : h \in H \}$. Put $r(H,K) = inf\{ r(H,k) : k \in K \}$ and let $C(H,K) = \{ k \in K : r(H,k) = r(H,K) \}$. When K is convex, we say that K has *normal structure* if for each bounded closed convex subset W of K with more than one point, there exists $x \in W$ such that $r(W,x) < \delta(W) = diam(W)$, or equivalently, $C(W,W)$ is a proper subset of W. DeMarr [1] showed that every compact convex subset of a

*Corresponding author: f.naderi@modares.ac.ir;

Banach space has normal structure, but Alspach [2] observed that this is not true for a weakly compact convex subset.

In [3] Kirk proved the following theorem:

Theorem 1.1. For any non-empty weakly compact convex subset K of a Banach space E with normal structure, each non-expansive self mappings T on K has a fixed point in K.

Kirk's theorem was a landmark in fixed point theory and inspired many subsequent works [4]. For example later on Lim [5] generalized Kirk's theorem for some special families of non-expansive self-maps on K according to the following considerations.

Let S be a *semi-topological semigroup*, i.e. S is a semigroup with a Hausdorff topology such that the multiplication is separately continuous, i.e. for each $a \in S$, the mappings $s \longrightarrow sa$ and $s \longrightarrow as$ from S into S are continuous. S is called *left reversible* if any two closed right ideals of S have non-void intersection.

An action of S on a topological space E is a mapping $(s,x) \to s(x)$ from $S \times E$ into E such that $(st)(x) = s(t(x))$ for all $s,t \in S, x \in E$. The action is separately continuous if it is continuous in each variable when the other is kept fixed. Every action of S on E induces a representation of S as a semigroup of self-mappings on E denoted by $\mathcal{S}$, and the two semigroups are usually identified. When the action is separately continuous, each member of $\mathcal{S}$ is a continuous mapping on E. A subset $K \subseteq E$ is S-invariant if $sK \subseteq K$ for each $s \in S$. We say that S has a common fixed point in E, if there exists a singleton S-invariant subset of E. When E is a normed space the action of S on E is *non-expansive* if $\|s(x) - s(y)\| \leq \| x - y\|$ for all $s \in S$ and $x, y \in E$.

Now we can state Lim's theorem [5]:

Theorem 1.2. Let K be a non-empty weakly compact convex subset of a Banach space E with normal structure and S be a left reversible semi-topological semigroup. Then every separately continuous and non-expansive action of S on K has a common fixed point in K.

One can also weaken the topology of set K in Lim's result as in [6,7] but here we focus on changing the structure of semigroup S.

Let $l^{\infty}(S)$ be the C*-algebra of all bounded complex-valued functions on S with supremum norm and point-wise multiplication. For each $s \in S$ and $f \in l^{\infty}(S)$, denote by $l_s(f)$ and $r_s(f)$ the left and right translates of f by s respectively, that is $l_s(f)(t) = f(st)$ and $r_s(f)(t) = f(ts)$ for all $t \in S$. Let X be a closed subspace of $l^{\infty}(S)$ containing constants and be invariant under translations. Then a linear functional $m \in X^*$ is called a *mean* if $\|m\| = m(1) = 1$, and a *left invariant mean* (LIM) if moreover $m(l_s(f)) = m(f)$ for $s \in S, f \in X$. Let $C_b(S)$ be the space of all bounded continuous complex-valued functions on S with supremum norm and $LUC(S)$ be the space of left uniformly continuous functions on S, i.e. all functions $f \in C_b(S)$ for which the mapping $s \to l_s f: S \longrightarrow C_b(S)$ is continuous when $C_b(S)$ has the sup-norm topology. Then $LUC(S)$ is a C*-subalgebra of $C_b(S)$ invariant under translations and containing constant functions. S is called *left amenable* if $LUC(S)$ has a LIM. The space of all right uniformly continuous functions, $RUC(S)$, and right amenability can be defined similarly. The semi-topological semigroup S is called *amenable* if it is both left and right amenable, in this situation there is a mean which is both left and right invariant. Left amenable semi-topological semigroups include commutative semigroups, as well as compact and solvable groups. The free (semi)group on two or more generators is not left amenable, however. For more details on amenability, examples and relations see [8,9,10,11,12,13,14].

A semigroup can be left amenable without being left reversible as the following example shows [12]:

Example 1.4. Let S be a topological space which is regular and Hausdorff. Then $C_b(S)$ consists of constant functions only. Define on S the multiplication $st = s$ for all $s, t \in S$. Let $a \in S$ be fixed. Define $\mu(f) = f(a)$ for all $a \in S$. Then μ is a left invariant mean on $C(S)$, but S is not left reversible.

Now the question naturally arises as to whether this is true if one considers a left amenable semi-topological semigroup in Lim's theorem instead of left reversible semi-topological semigroup? In this paper, we show that the answer is affirmative. Our theorem is new and not a result of any previous work.

2 Main Theorem

The space of almost periodic functions on S, is the space of all $f \in C(S)$ such that $\{ l_s f: s \in S \}$ is relatively compact in the sup-norm topology of $C(S)$ and is denoted by $AP(S)$. Also the space of all $f \in C(S)$ such that $\{ l_s f: s \in S \}$ is relatively compact in the weak topology of $C(S)$ is denoted by $WAP(S)$ and called the space of weakly almost periodic functions on S. For any semi-topological semigroup S we have the following theorem ([8], p.131 and p.164):

Theorem 2.1. For a semi-topological semigroup S the following assertions hold:

a) $f \in AP(S)$ if and only if $\{ r_s f: s \in S \}$ is relatively compact in the sup-norm topology of $C(S)$.
b) $f \in WAP(S)$ if and only if $\{ r_s f: s \in S \}$ is relatively compact in the weak topology of $C(S)$.
c) $AP(S) \subseteq LUC(S) \cap RUC(S)$.

A topological semigroup is a semi-topological semigroup with jointly continuous multiplication. Let S^a (respectively S^w) be the almost periodic (respectively weakly almost periodic) compactification of S, i.e. S^a (respectively S^w) be the spectrum of C*-algebra $AP(S)$ (respectively $WAP(S)$). Then S^a and S^w are semi-topological semigroups with the multiplication given by: $< m.n, f > = < m, n.f >$, where $n.f(s) = < n, l_s f >, m, n \in S^a$ (respectively S^w), $f \in AP(S)$ (respectively $WAP(S)$). The following theorem can be found in ([10], p.7).

Theorem 2.2. If S^w is a topological semigroup, then $AP(S) = WAP(S)$.

The action of a semigroup S on a weakly compact Hausdorff space M is equicontinuous if, for each $y \in M$ and $U \in \mathcal{U}$, where $\mathcal{U}$ is the unique uniformity which determines the topology of M (see [15], p.197), there is a $V \in \mathcal{U}$ such that $(sx, sy) \in U$ for all $s \in S$ whenever $(x, y) \in V$.

Lemma 2.3. Let S be a semi-topological semigroup with separately continuous, non-expansive and equicontinuous action on a weakly compact subset M of a Banach space E. Let $m \in M$, $f \in C(M)$ and define $f_m(s) = f(sm)$ $(s \in S)$. If S^w is a topological semigroup, then $f_m \in LUC(S)$.

Proof: Put pointwise convergence topology on $C(S)$. We claim that the mapping: $M \longrightarrow C(S)$ defined by $m \longrightarrow f_m$ is continuous. Obviously f_m is in $C(S)$, since f is continuous and the action is separately continuous. Let $m_\alpha \longrightarrow m$ and βS be the Stone-Cech compactification of S. We show that the net (f_{m_α}) converges pointwisely to f_m, i.e. $f_{m_\alpha}(\mu) \longrightarrow f_m(\mu)$ for any $\mu \in \beta S$. The element μ is not necessarily in S, so we must find a suitable convergent net and relate its convergence to the convergence of net $(f_{m_\alpha}(\mu))$ in some ways. Note that the Stone-Cech compactification of S is precisely the spectrum of C*-algebra $C(S)$ and it is well-known that the set of all point-mass measures is weak*-dense in the spectrum of $C(S)$, so there must be a net (s_α) in S such that the corresponding net (δ_{s_α}) of point-mass measures converges to μ in the weak*-topology of the dual space. Hence $\delta_{s_\alpha}(g) \longrightarrow \mu(g)$ for any $g \in C(S)$, especially for $f_m \in C(S)$ we have

$$f(s_\alpha m) = f_m(s_\alpha) = \int f_m \, d\delta_{s_\alpha} = \delta_{s_\alpha}(f_m) \longrightarrow \mu(f_m)$$

On the other hand, (s_α) is a net of self-mappings on M and converges pointwisely to a continuous self-mapping t on M, by equicontinuity of the action. So $s_\alpha m \longrightarrow tm$ in M, and then $f(s_\alpha m) \longrightarrow f(tm)$ by continuity of f. Now the net $(f(s_\alpha m))$ converges to limits $f(tm)$ and $\mu(f_m)$, hence $f(tm) = \mu(f_m)$.

Now $\mu(f_{m_\alpha}) = f(tm_\alpha) \longrightarrow f(tm) = \mu(f_m)$, but $\mu(f_m) = f_m(\mu)$ and $\mu(f_{m_\alpha}) = f_{m_\alpha}(\mu)$ by the very definition of Gelfand transform. Therefore $f_{m_\alpha}(\mu) \longrightarrow f_m(\mu)$, which proves the claim.

For each right translate of f_m we have

$$r_a(f_m)(s) = f_m(sa) = f(sam) = f_{am}(s); \quad (s, a \in S)$$

hence $\{ r_a f_m : a \in S\} = \{ f_{am} : a \in S\} = f_{Sm}$. The set Sm is relatively compact in M and the set f_{Sm} is the image of Sm under the continuous mapping $m \longrightarrow f_m$, so f_{Sm} is relatively compact in the pointwise convergence topology of $C(S)$. Now by ([8], theorem 4.2.3) and theorem 2.1 part (b) we see that $f_m \in WAP(S)$, but according to theorem 2.2 $WAP(S) = AP(S)$. Applying theorem 2.1 this time part (c), we see that $f_m \in LUC(S)$.

Now we use the above lemma to prove our theorem:

Theorem 2.4. Let K be a non-empty weakly compact convex subset of a Banach space E with normal structure and S be a left amenable semi-topological semigroup for which S^w is a topological semigroup. Then every separately continuous, non-expansive and equicontinuous action of S on K has a common fixed point in K.

Proof: An application of Zorn's lemma shows that there exists a minimal non-empty weakly compact convex and S-invariant subset $X \subseteq K$. If X is a singleton we are done, otherwise apply Zorn's lemma for the second time to get a minimal non-empty weakly compact and S-invariant subset $M \subseteq X$.

If M is singleton we are done, otherwise if $\delta(M) = diam(M) > 0$, we get a contradiction by normal structure assumption of K which implies that

$$\exists\, u \in \overline{co}(M) \; such\ that\ r_0 = sup\{ \|m - u\| : m \in M\} < \delta(M).$$

Let $X_0 = \bigcap_{m \in M} B[m, r_0] \cap X$. Then X_0 is a non-empty (indeed $u \in X_0$) convex proper subset of X. The set X_0 is weakly compact, since every closed convex ball is weakly compact. To arrive at a contradiction we need to show that X_0 is S-invariant, i.e. $sX_0 \subseteq X_0$ for each s in S.

To this end we show that M is S-preserved, i.e. $sM = M$ for all $s \in S$. Let ν be a left invariant mean on $LUC(S)$ and define $\mu(f) = \nu(f_m)$, where f_m is defined as in lemma 2.3. By Riesz representation theorem, μ induces a regular probability measure on M (still denoted by μ) such that $\mu(sB) = \mu(B)$ for all Borel sets $B \subseteq M$ and $s \in S$. Let F be the support of μ. Each $s \in S$ defines a measurable continuous function from M into M, so by basic properties of the support, $F \subseteq sM$ and $\mu(sM) = \mu(M) = 1$. Assume that χ_F is the characteristic function of F. For each $s \in S$,

$$1 = \mu(F) = \int_M \chi_F(y)\, d\mu = \int_M \chi_F(sy)\, d\mu = \mu(s^{-1}F),$$

($s^{-1}F$ means the pre-image of F under s) again by the definition and properties of support we see that $F \subseteq s^{-1}F$, meaning that F is S-invariant. Hence $F = M$ by the minimality of M. Consequently $M = F \subseteq sM$ for each $s \in S$. But M was already S-invariant, so $sM = M$ for each s in S.

Let $x \in X_0, s \in S$ and $m \in M$ be arbitrary. The element s defines an onto mapping on M, so there must be an m' such that $m = sm'$. Hence

$$\|sx - m\| = \|sx - sm'\| \leq \|x - m'\| \leq r_0$$

which show that X_0 is S-invariant. Therefore we conclude that X_0 is a proper subset of X with the same properties as, but this contradicts the minimality of X. So M contains only one point which is a common fixed point for the action of S.

Here is an application of our theorem:

Example 2.5. Let $K = \{(r, \theta): 0 \leq r \leq 1,\ 0 \leq \theta < 2\pi\}$ be the closed unit disc in $\mathbb{R}^2$ in polar coordinates. It is well-known that K is convex, weakly compact and has normal structure. Define the self mappings f and g on K by $f(r, \theta) = (r, \pi + \theta)$ and $g(r, \theta) = (r, -\theta)$. Let S be the discrete semigroup generated by f and g under composition. Then S is a finite group, hence amenable. Also $S^w = S$, showing that S^w is a topological semigroup. The action of S is non-expansive, since f is a rotation and g is a reflection and according to the elementary geometry, rotations and reflections are isometries in the plane. On the other hand $\mathcal{U} = \{ U(\varepsilon): \varepsilon > 0 \}$, where $U(\varepsilon) = \{ (x, y) \in K \times K : \|x - y\| < \varepsilon\}$, is the unique uniformity which determines the topology of K. Suppose that $y \in K$ and $U = \{ (x, y) \in K \times K : \|x - y\| < \varepsilon\}$ be in $\mathcal{U}$, by Archimedean property of real numbers there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. Let $V = \{ (x, y) \in K \times K : \|x - y\| < \frac{1}{n}\}$. Then for any $(x, y) \in V$ and $\in S$, using the non-expansive property of the action we have

$$\|sx - sy\| \leq \|x - y\| < \frac{1}{n} < \varepsilon,$$

so $(sx, sy) \in U$. Hence the action of S is equicontinuous. Now theorem 2.4 is applicable and predicts a fixed point for the action of S on K. For example origin is a fixed point for this action, i.e. rotations and reflections do not change the position of the center of symmetry.

3 Conclusion

In this paper we have shown that when a left amenable semi-topological semigroup S acts on a non-empty weakly compact convex subset K of a Banach space E with normal structure such that the action is separately continuous, non-expansive and equicontinuous then there is a fixed point for this action provided that the weak almost periodic compactification of S is a topological semigroup. It is an open question if the condition of being topological semigroup for S^w in our theorem can be dropped or not?

Competing Interests

Author has declared that no competing interests exist.

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