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A New Technique for Solving Black-Scholes Equation for Vanilla Put Options

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Abstract

In this paper we present a new technique for solving Black-Scholes equation for vanilla put options using the Mellin transform method.

Keywords: Black-Scholes equation; Mellin transform method; vanilla put option.

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1 Introduction

Vanilla option is a contract that gives the holder the right, but not the obligation, to buy or sell an instrument at a specified strike price on or before a specified date. The common examples of vanilla options are American and European options.

American options are options that allow option holders to exercise the option at any time prior to and including its maturity date, thus increasing the value of the option to the holder relative to European options, which can only be exercised at maturity. The majority of exchange-traded options are American [1].

Nowadays, the Black-Scholes equation derived by [2] is widely used in the field of mathematical finance. Despite the success of the Black-Scholes model on hedging and pricing contingent claims, [3] noted early that options quoted on the markets differ systematically from their predicted values, which led up to questioning the distributional assumptions based on geometric wiener process [4]. Alternative approach for

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the solution of the Black-Scholes partial differential equation for European call option which pays dividend yield was considered by [4]. [5] introduced the Mellin transforms in the theory of option pricing. They derived the expression for the free boundary and price of an American perpetual put as the limit of finite lived options. In this paper we shall derive a new technique which does not require transformation of variables for solving Black-Scholes equation for vanilla put options using the Mellin transforms. See ([6-17]) for the mathematical backgrounds of the Mellin transform method for options valuation and other approaches for the solution of Black-Scholes equation.

The rest of the paper is structured as follows. Section 2 introduces the Mellin transforms. In section 3 we present a new technique for solving Black-Scholes equation for vanilla put options such as European put options. Section 4 concludes the paper.

2 The Mellin Transforms

The Mellin transform denoted by M is a complex valued function defined on a vertical strip in the v-plane and is given by

$$M(f(y),v) \equiv F(v) = \int_0^\infty f(y)y^{v-1}dy \tag{1}$$

where $f(y)$ is a function defined on the positive real axis $y \in (0,\infty)$. The function $F(v)$ is called the Mellin transform of $f(y)$. The Mellin transform of $f(y)$ exists for any $v \in C$ on $-a_1 < R(v) < -a_2$ where $f(y)$ is defined as

$$f(y) = \begin{cases} O(y^{a_1}),\ y \to 0 \\ O(y^{a_2}),\ y \to \infty \end{cases}$$

Conversely, the Mellin inversion formula of (1) is defined as

$$M^{-1}(F(v)) \equiv f((y),v) = \frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty} F(v)y^{-v}dv \tag{2}$$

where the integration is along a vertical line through $\text{Re}(v) = a$.

Some of the basic fundamental properties of the Mellin transforms are detailed below.

(a) Shifting Property

$$M\left(y^a f(y),v\right) = \int_0^\infty y^a f(y)y^{v-1}dy = F(v+a) \tag{3}$$

(b) Scaling Property

$$M\left(f(ay),v\right) = \int_0^\infty f(ay)y^{v-1}dy = a^{-v}F(v) \tag{4}$$

(c) The Mellin Transform of Derivatives

$$M\left(\frac{d^k}{dy^k}f(y),v\right)=(-1)^k(v-k)_k F(v-k),\ (v-k)\in V_f,\ k\in Z^+ \tag{5}$$

where the symbol $(v-k)_k$ is defined for k integer by;

$$(v-k)_k=(v-k)(v-k+1)...(v-1)$$

Some important results of the Mellin transforms are given by

$$\begin{cases} M\left(\left(v\frac{d}{dy}\right)^k f(y),v\right)=(-1)^k v^k F(v) \\ M\left(\frac{d^k}{dv^k}v^k f(y),v\right)=(-1)^k(v-k)_k F(v) \\ M\left(v^k\frac{d^k}{dv^k}f(y),v\right)=(-1)^k(v)_k F(v) \end{cases} \tag{6}$$

where $k\in Z^+$ and $v_k=v(v+1)...(v+k-1)$.

3 New Technique for the Solution of the Black-Scholes Equation for Vanilla Put Options

Let us consider the Black-Scholes equation for vanilla put options given by

$$\left.\frac{\partial P(S_t,t)}{\partial t}+\frac{\sigma^2 S_t^2}{2}\frac{\partial^2 P(S_t,t)}{\partial S_t^2}+rS_t\frac{\partial P(S_t,t)}{\partial S_t}-rP(S_t,t)=0,\ S_t\in(0,\infty),t\in[0,T]\right\} \tag{7a}$$

with the boundary conditions

$$\left.\begin{array}{ll} P(S_t,T)=\max\left[\left(K-S_t\right),0\right], & \text{on } [0,\infty) \\ \lim_{S_t\to 0}P(S_t,t)=K\exp(-r(T-t)), & \text{on } [0,T) \\ \lim_{S_t\to\infty}P(S_t,t)=0, & \text{on } [0,T) \end{array}\right\} \tag{7b}$$

where σ is the volatility, r is the risk-free interest rate, K is the strike price and T is the maturity date. It is a known fact that (7a) has a closed form solution obtained after several change of variables and solving certain related diffusion equations. This procedure is not applicable in the vector framework where $P(S_t,t)$ is a vector and σ, r are matrices.

The Mellin transform for the price of vanilla put options is defined as

$$M(P(S_t,t),v) = p(v,t) = \int_0^{\infty} P(S_t,t) S_t^{v-1} dS_t \tag{8}$$

The Mellin inversion formula of (8) is given by

$$M^{-1}(p(v,t)) = P((S_t,t),v) = \frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty} p(v,t) S_t^{-v} dv \tag{9}$$

To use the Mellin transform and the conditions that guarantee its existence, we assume that $p(S_t,t)$ is bounded of polynomial degree when $S_t \to 0$ and $S_t \to \infty$ i.e.

$$p(S_t,t) = \begin{cases} O(S_t^{a_1}), S_t \to 0 \\ O(S_t^{a_2}), S_t \to \infty \end{cases} \tag{10}$$

for any $v \in C$ on $-a_1 < R(v) < -a_2$ where $(-a_1, -a_2)$ is called fundamental strip.

Taking the Mellin transform of the Black-Scholes equation for vanilla put option given by (7a), we have that

$$M\left(\frac{\partial P(S_t,t)}{\partial t} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 P(S_t,t)}{\partial S_t^2} + rS_t\frac{\partial P(S_t,t)}{\partial S_t}\right) = M\left(rP(S_t,t)\right) \tag{11}$$

Using the properties of the Mellin transforms, we have

$$\left.\begin{aligned} &M\left(\frac{\partial P(S_t,t)}{\partial t}\right) = \frac{d}{dt}p(v,t) \\ &M\left(\frac{\sigma^2 S_t^2}{2}\frac{\partial^2 P(S_t,t)}{\partial S_t^2}\right) = \frac{\sigma^2}{2}(v^2+v)p(v,t) \\ &M\left(rS_t\frac{\partial P(S_t,t)}{\partial S_t}\right) = -rvp(v,t) \\ &M\left(rP(S_t,t)\right) = rp(v,t) \end{aligned}\right\} \tag{12}$$

Substituting (12) into (11) and simplifying further yields

$$\frac{dp(v,t)}{dt} = -\left(\frac{v^2\sigma^2}{2} + \left(\frac{\sigma^2}{2} - r\right)v - r\right)p(v,t), \quad t \in [0,T] \tag{13}$$

Integrating (13) and solving further yields

$$p(v,t) = A(v)\exp\left(-\left(\frac{v^2\sigma^2}{2} + \left(\frac{\sigma^2}{2} - r\right)v - r\right)\right)t \tag{14}$$

Let $\varphi(v) = \left(\frac{v^2\sigma^2}{2} + \left(\frac{\sigma^2}{2} - r\right)v - r\right)$, then (14) becomes

$$p(v,t) = A(v)\exp\left(-\varphi(v)\right)t \tag{15}$$

where $A(v)$ is a constant of integration denoted by

$$A(v) = \psi(v,t)\exp(\varphi(v)T) \tag{16}$$

$\psi(v,t)$ is obtained by taking the Mellin transform of the initial condition given by

$$P(S_t,T) = \theta(S_t,v) = \max\left(\left(K - S_t\right),0\right) \tag{17}$$

then we have

$$\psi(v,t) = \int_0^\infty \max(K - S_t,0)^+ S_t^{v-1} dS_t = \frac{K^{v+1}}{v(v+1)} \tag{18}$$

Using equations (15), (16) and (18), we have that

$$p(v,t) = \frac{K^{v+1}}{v(v+1)}\exp(-\varphi(v)(t-T)) \tag{19}$$

The Mellin inversion of (19) is obtained as

$$M^{-1}(p(v,t)) = P(S_t,t) = \frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty} \frac{K^{v+1}}{v(v+1)}\exp(-\varphi(v)(t-T))S_t^v dv \tag{20}$$

Next shall show that (20) is a solution of the Black-Scholes equation for vanilla put options given by (7a). Following the procedures in [4], we assume that

$$v = m + jn \quad \Rightarrow dv = jdn, \text{ where } v \in C \tag{21}$$

where *m* and *n* are the real and the imaginary parts of the complex number $v \in C$.

Substituting (21) into (20) yields

$$P(S_t,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} S_t^{-(m+jn)} \exp(-\varphi(m+jn)(t-T))dn \tag{22}$$

Substituting $v = m + jn$ into $\varphi(v) = \left(\frac{v^2\sigma^2}{2} + \left(\frac{\sigma^2}{2} - r\right)v - r\right)$, then

$$\left.\begin{aligned} -\varphi(m+jn) &= -\left(\frac{(m+jn)^2\sigma^2}{2} + \left(\frac{\sigma^2}{2} - r\right)(m+jn) - r\right) \\ &= \left(-\frac{\sigma^2 m^2}{2} + \frac{\sigma^2 n^2}{2} - \frac{\sigma^2 m}{2} + rm + r\right) + j\left(-mn\sigma^2 - \frac{\sigma^2 n}{2} + rn\right) \end{aligned}\right\} \quad (23)$$

Since $P(S_t, t)$ is continuous and Mellin transformable, setting $t = T$ then (22) becomes

$$P(S_t, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} S_t^{-(m+jn)} dn \quad (24)$$

Equation (22) satisfies (24) and it is well defined. Using the definition of the Mellin transforms, then we have

$$\left|\frac{K^{1+m+jn}}{(m+jn)(m+jn+1)}\right| \le M(m) = \int_0^\infty \left|f(s)\right| s^{m-1} ds \quad \forall m \in R \quad (25)$$

and for $t \in [0, T)$ we have that

$$\left\{\begin{aligned} &\int_{-\infty}^{\infty} \left|\frac{K^{1+m+jn}}{(m+jn)(m+jn-1)}\right| \left|S_t^{-(m+jn)}\right| \left|\exp(\varphi(m+jn)(T-t))\right| dn \\ &\le M(m) S_T^{-m} \exp\left(\left(-\frac{\sigma^2 m^2}{2} - \frac{\sigma^2 m}{2} + rm + r\right)(t-T)\right) \int_{-\infty}^{\infty} \exp\left(\left(\frac{-\sigma^2 n^2}{2}\right)(T-t)\right) dn \end{aligned}\right. \quad (26)$$

Using the differentiation theorem of parametric integrals and the fact that

$$\int_{-\infty}^{\infty} n^j \exp\left(-\frac{\sigma^2 n^2}{2}\right)(T-t) dn < \infty, \quad j = 0, 1, 2, \ldots, t \in [0, T) \quad (27)$$

Then it follows that upon differentiation of (22), one gets

$$\left.\begin{aligned} \frac{\partial P(S_t, t)}{\partial t} &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} \varphi(m+jn) S_t^{-(m+jn)} \exp(-\varphi(m+jn)(t-T)) dn \\ \frac{\partial P(S_t, t)}{\partial S_t} &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} (m+jn) S_t^{-(m+jn+1)} \exp(\varphi(m+jn)(t-T)) dn \end{aligned}\right\} \quad (28)$$

$$\frac{\partial^2 P(S_t,t)}{\partial S_t^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)}(m+jn)(m+jn+1)S_t^{-(m+jn+2)} \exp(-\varphi(m+jn)(t-T))dn \tag{29}$$

Substituting (22), (28) and (29) into (7a), we have that

$$\left.\begin{aligned}
&\frac{\partial P(S_t,t)}{\partial t} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 P(S_t,t)}{\partial S_t^2} + rS_t\frac{\partial P(S_t,t)}{\partial S_t} - rE_c(S_t,t) = \\
&-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)}\varphi(m+jn)S_t^{-(m+jn)} \exp(-\varphi(m+jn)(t-T))dn + \\
&\frac{\sigma^2 S_t^2}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)}(m+jn)(m+jn+1)S_t^{-(m+jn+2)} \exp(\varphi(m+jn)(t-T))dn - \\
&rS_t\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)}(m+jn)S_t^{-(m+jn+1)} \exp(-\varphi(m+jn)(t-T))dn + \\
&r\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)}S_t^{-(m+jn)} \exp(\varphi(m+jn)(t-T))dn \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-\varphi(m+jn) + \frac{\sigma^2 S_t^2}{2}(m+jn)(m+jn+1)S_t^{(-2)} - rS_t(m+jn)S_t^{(-1)} - r\right) \\
&\quad\times\left(\frac{K^{1+m+jn}}{(m+jn)(m+jn+1)}S_t^{-(m+jn)} \exp(-\varphi(m+jn)(t-T))\right)dn \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-\varphi(m+jn) + \left(\left(\frac{\sigma^2 m^2}{2} - \frac{\sigma^2 n^2}{2} + \frac{\sigma^2 m}{2} - rm - r\right) + j\left(mn\sigma^2 + \frac{\sigma^2 n}{2} - rn\right)\right)\right) \\
&\quad\times\frac{K^{1+m+jn}}{(m+jn)(m+jn+1)}S_t^{-(m+jn)} \exp(-\varphi(m+jn)(t-T))dn \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-\varphi(m+jn) + \varphi(m+jn)\right)\frac{K^{1+m+jn}}{(m+jn)(m+jn+1)}S_t^{-(m+jn)} \exp(\varphi(m+jn)(t-T))dn \\
&= 0
\end{aligned}\right\} \tag{30}$$

Hence the expression for the price of vanilla put options denoted by $P(S_t,t)$ and given by (20) is a solution of (7a).

4 Conclusion

In this paper we presented a new technique for solving Black-Scholes equation for vanilla put options using the Mellin transform method. The Mellin transform method is a powerful technique that can also be used for the valuation of more complex vanilla options and some path dependent options and can be extended to jump diffusion processes, stochastic volatility and other stochastic processes.

Competing Interests

Authors have declared that no competing interests exist.

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