An Effective Java-based Algorithm to Compute Eigenvalues of Symmetric Toeplitz Matrices

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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ABSTRACT

Eigenvalues are a special set of scalars associated with a linear system of equations, that are sometimes also known as characteristic roots. In this work, a Java-based algorithm is developed to find the eigenvalues for Toeplitz matrices using Adomian's decomposition method (shortly, ADM). Illustrative examples will be examined to support the proposed analysis.

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1 INTRODUCTION

Many interesting phenomena in scientific and engineering applications are governed by algebraic equations. It is also known that there are two types of these equations, the linear, or nonlinear equations, and could be system. Most of these types of equations do not have an analytical solution, these equations should be solved by using numerical or approximate methods. In using numerical methods, like Sinc method, the solution reduces to the discrete system $A\mathbf{x} = \mathbf{b}$, in which the coefficient matrix A is a combinations of Toeplitz matrices and diagonal matrices. Hence, it is a basic requirement to discuss the algebraic properties of these Topeplitz matrices. Toeplitz matrices arise in a variety of applications in Mathematics and Engineering. In particular, when the Sinc method is applied to discretize the differential equation (Ordinary, partial or integral), we can often obtain a linear system whose coefficients matrices are combinations of Toeplitz and diagonal matrices, see [1, 2]. One of the main advantages of using the technique in this paper is discussing the stability of the discrete system being obtained by Sinc methodology, which is considered to be a basic requirement in finding bounds for the eigenvalues of Sinc matrices.

In the last decade, there has been some advanced developments including, Adomian decomposition method [3, 4, 5, 6], Differential transform method [7], and Homotopy perturbation method [8] for solving various types algebraic equations. Eigenvalues are a special set of scalars associated with a linear system of equations, that are sometimes also known as characteristic roots. The determination of the eigenvalues of a system, like $A\mathbf{x} = \mathbf{b}$, is important in physics and engineering, where it is equivalent to matrix diagonalization, and arises in such common applications as, stability analysis, the physics of rotating bodies, and small oscillations of vibrating systems.

The basic motivation of this work, is to propose a new modification of the ADM to find eigenvalues for any $n \times n$ matrix. Therefore, as mentioned above, the objective of this paper is to find eigenvalues for Toeplitz matrices that comes from the theory of sinc function. At the beginning of the 80s, a new method, called ADM for solving various kinds of nonlinear equations had been proposed by Adomian [9, 10]. The convergence of Adomian's method has been investigated by several authors [11]. ADM offers a reasonable, reliable solution to algebraic equations. The system is implemented for the full Java language, and is used to statically verify the correctness of Java. To demonstrate this we intend to solve two examples in the succeeding sections considering the symmetry of the given matrix. The outlines of the paper is as follows. In section 2, we derive the expression of the Toeplitz matrices from sinc function. The basic ideas of the ADM for solving algebraic equation is reviewed in Section 3. Numerical experiments are presented in section 4. Finally we give a concluding remark on the applications of the method.

2 TOEPLITZ MATRICES

We first give general expressions for the Toeplitz matrices associated with the Sinc discretization of various order. The goal of this section is to recall notations and definitions of the Sinc function that will be used in this paper. These are discussed in [1, 2]. The Sinc function is defined on the whole real line $\mathbb{R}$ by

$$\text{sinc}\,(x) = \frac{\sin(\pi x)}{\pi x}, x \in \mathbb{R}. \qquad (2.1)$$

the k-th Sinc function is defined as

$$S(k,h)(x) = \text{sinc}\,[(x-kh)/h] = \frac{\sin[\pi(x-kh)/h]}{\pi(x-kh)/h}$$

The properties of Sinc functions have been extensively studied in [1, 2]. The sinc method requires that the derivatives of sinc functions be evaluated at the nodes. Technical calculations provide the following results that will be useful in formulating the discrete system [1, 2], and these quantities are delineated by $\delta_{jk}^{(q)} = h^q \frac{d^q}{d\phi^q}[S_j \circ \phi(x)]\Big|_{x=x_k}, q = 0, 1, 2.$

In particular, the following convenient notation will be useful in formulating the discrete system

$$\delta_{jk}^{(0)} = \left.[S(j,h) \circ \phi(x)]\right|_{x=x_k} = \begin{cases} 1, & j = k \\ 0, & j \neq k, \end{cases}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}\left.[S(j,h) \circ \phi(x)]\right|_{x=x_k} = \begin{cases} 0, & j = k \\ \frac{(-1)^{k-j}}{(k-j)}, & j \neq k \end{cases}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}\left.[S(j,h) \circ \phi(x)]\right|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j = k \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k \end{cases}$$

and,

$$\delta_{jk}^{(3)} = h^3\frac{d^3}{d\phi^3}\left.[S(j,h) \circ \phi(x)]\right|_{x=x_k} = \begin{cases} 0, & j = k \\ \frac{(-1)^{k-j}[6-(k-j)^2\pi^2]}{(k-j)^3}, & j \neq k \end{cases}$$

Then define the $m \times m, (m = 2N+1)$ Toeplitz matrices $I_m^{(q)} = [\delta_{jk}^{(q)}], q = 0,1,2,3,...$ i.e., the matrix whose $jk-$ entry is given by $\delta_{jk}^{(q)}, q = 0,1,2,3,...$ Also define the diagonal matrix $\mathcal{D}(g) = \text{diag}[g(x_{-N}),...,g(x_N)]$. Note that the matrix $I^{(2)}$ is a symmetric matrix, i.e., $I_{jk}^{(2)} = I_{kj}^{(2)}$. The matrix $I^{(1)}$ is skew-symmetric matrix, i.e., $I_{jk}^{(1)} = -I_{kj}^{(1)}$ and $I^{(3)}$ takes the form

$$I_m^{(3)} = \begin{pmatrix} 0 & -(6-\pi^2) & \frac{6-2^2\pi^2}{2^3} & \cdots & \frac{(-1)^{m-1}[6-(m-1)^2\pi^2]}{(m-1)^3} \\ 6-\pi^2 & 0 & \ddots & \ddots & \vdots \\ \frac{-(6-2^2\pi^2)}{2^3} & 6-\pi^2 & \ddots & -(6-\pi^2) & \frac{6-2^2\pi^2}{2^3} \\ \vdots & \ddots & \ddots & 0 & -(6-\pi^2) \\ \frac{(-1)^m[6-(m-1)^2\pi^2]}{(m-1)^3} & \cdots & \frac{-(6-2^2\pi^2)}{2^3} & 6-\pi^2 & 0 \end{pmatrix}, \quad (2.2)$$

$$I_m^{(2)} = \begin{pmatrix} \frac{-\pi^2}{3} & 2 & \vdots & \frac{-2(-1)^{m-1}}{(m-1)^2} \\ 2 & \cdot & \cdot & \cdot \\ \vdots & \cdot & \cdot & \vdots \\ \frac{-2(-1)^{m-1}}{(m-1)^2} & \vdots & 2 & \frac{-\pi^2}{3} \end{pmatrix}, \quad (2.3)$$

$$I_m^{(1)} = \begin{pmatrix} 0 & -1 & \cdots & \frac{(-1)^{m-1}}{m-1} \\ 1 & 0 & & \vdots \\ \vdots & & & \vdots \\ \frac{(-1)^{m-1}}{m-1} & \cdots & 1 & 0 \end{pmatrix} \quad (2.4)$$

Let A be an $n \times n$ matrix, if a vector $\underline{x} \in \mathbb{R}^n, (\underline{x} \neq \mathbf{0})$ such that $A\underline{x} = \lambda\underline{x}$ for some scalar λ is called the eigenvalue of the matrix A with corresponding eigenvector $\underline{x}$. With λ as an eigenvalue of A, we need to solve

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \vdots & \ldots & & \vdots \\ a_{n1} & \ldots & \ldots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (2.5)$$

which can be written as $(A - \lambda I)\underline{\mathbf{x}} = \mathbf{0}$, where I is the identity matrix. To solve (2.5), we need to find $det(A - \lambda I) = 0$. The determinant on the right hand of the previous equation can be expanded to give what is called the characteristic equation of A.

$$p_A(\lambda) = \lambda^n - trac(A)\lambda^{n-1} + (\text{Polynomial of degree} n - 2)$$

which can be written in more general form as

$$p_A(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-1}\lambda + a_n \tag{2.6}$$

The eigenvalues of A are those numbers λ for which $p_\lambda(x) = 0$. In general, from the fundamental theorem of algebra, there are n of these, in which our main goal in this paper is find these numbers. In most cases it is difficult to obtain an analytical solution of (2.6). Therefore the exploitation of numerical techniques for solving such equations becomes a main subject of considerable interests. Probably the most well-known and widely used algorithm to find a roots of equation (2.6) is Newton's method [5].

3 BASIC IDEA OF THE ADM

We apply the Adomian decomposition method (ADM) to find the smallest eigenvalue for a given matrix, for that we solve the obtained characteristic equation via the use of ADM [9, 10]. To illustrate the basic idea of this method.

$$a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-1}\lambda + a_n = 0, \tag{3.1}$$

Which can be written in a fixed-point form, via $\lambda = G(\lambda)$. Provided that $a_{n-1} \neq 0$ as

$$\lambda = -\frac{a_0}{a_{n-1}}\lambda^n - \frac{a_1}{a_{n-1}}\lambda^{n-1} - ... - \frac{a_{n-2}}{a_{n-1}}\lambda^2 - \frac{a_n}{a_{n-1}} \tag{3.2}$$

The standard Adomian decomposition method uses the solution λ in terms of the series

$$\lambda = \sum_{j=0}^{\infty} \lambda_j \tag{3.3}$$

and the nonlinear functions $\lambda^2, \lambda^3, ..., \lambda^n$ are expressed in terms of an infinite series called Adomian's polynomials, such that for any nonlinear function $G(\lambda)$, we have

$$G(\lambda) = \sum_{j=0}^{\infty} A_j \tag{3.4}$$

where A_j are called the Adomian's polynomials obtained by the traditional formula

$$A_j = A_j(\lambda_0, \lambda_1, ..., \lambda_j) = \frac{1}{j!}\frac{d^j}{d\mu^j}G\Big(\sum_{k=0}^{\infty} \lambda_k\mu^k\Big)\Big|_{\mu=0} \tag{3.5}$$

where μ is the parameter introduced for convenience. Given a nonlinear function $G(\lambda)$, the first few Adomian's polynomials are given by

$$A_0 = G(\lambda_0),\ A_1 = \lambda_1 G'(\lambda_0),\ \ A_2 = \lambda_2 G'(\lambda_0) + \frac{1}{2!}\lambda_1^2 G''(\lambda_0),$$

and

$$A_3 = \lambda_3 G'(\lambda_0) + \lambda_1\lambda_2 G''(\lambda_0) + \frac{1}{3!}\lambda_1^3 G'''(\lambda_0)$$

Wazwaz [3] developed a new algorithm for calculating Adomian polynomials for all forms of nonlinearity. In this paper we are dealing with nonlinear polynomials only, so if $G(\lambda) = \lambda^2$, then set $\lambda = \sum_{n=0}^{\infty} \lambda_n$, so that

$$G(\lambda) = (\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + ...)^2$$

Expanding the expression at the right hand side gives

$$G(\lambda) = \lambda_0^2 + 2\lambda_0\lambda_1 + 2\lambda_0\lambda_2 + \lambda_1^2 + 2\lambda_0\lambda_3 + 2\lambda_1\lambda_2 + ...$$

rearrange the above terms by grouping all terms with the sum of the subscript of the components of λ_n is the same, we obtain

$$G(\lambda) = \underbrace{\lambda_0^2}_{A_0} + \underbrace{2\lambda_0\lambda_1}_{A_1} + \underbrace{2\lambda_0\lambda_2 + \lambda_1^2}_{A_2} + \underbrace{2\lambda_0\lambda_3 + 2\lambda_1\lambda_2}_{A_3} + ...$$

Similarly, if $G(\lambda) = \lambda^3$, then

$$G(\lambda) = (\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + ...)^3$$

Expanding the expression at the right hand side, and rearrange terms, by grouping all terms with the sum of the subscript of the components of λ_n is the same, we obtain

$$G(\lambda) = \underbrace{\lambda_0^3}_{A_0} + \underbrace{3\lambda_0^2\lambda_1}_{A_1} + \underbrace{3\lambda_0^2\lambda_2 + 3\lambda_0\lambda_1^2}_{A_2} + \underbrace{3\lambda_0^2\lambda_3 + 6\lambda_0\lambda_1\lambda_2 + \lambda_1^3}_{A_3} + ...$$

Now back to the procedure, upon substituting equations (3.3) and (3.4) into equation (3.2) we arrive at

$$\sum_{j=0}^{\infty} \lambda_j = -\frac{a_n}{a_{n-1}} - \frac{a_0}{a_{n-1}} \sum_{j=0}^{\infty} A_{n,j} - \frac{a_1}{a_{n-1}} \sum_{j=0}^{\infty} A_{n-1,j} - ... - \frac{a_{n-2}}{a_{n-1}} \sum_{j=0}^{\infty} A_{2,j} \qquad (3.6)$$

where $A_{n,j}, A_{n-1,j}, ..., A_{2,j}$ represent the Adomian polynomials for the nonlinear functions $\lambda^n, \lambda^{n-1}, ..., \lambda^2$ respectively. To determine the components $\lambda_n, n \geq$, we first identify the zeroth component λ_0 by all terms that are constant in equation (3.2). The remaining components of the series (3.3) can be determined in a way that each component is determined by using the preceding components, i.e., each term of the series (3.3) is given by by the following recursive relation

$$\begin{aligned} \lambda_0 &= -\frac{a_n}{a_{n-1}} \\ \lambda_{j+1} &= -\frac{a_0}{a_{n-1}} A_{n,j} - \frac{a_1}{a_{n-1}} A_{n-1,j} - ... - \frac{a_{n-2}}{a_{n-1}} A_{2,j}, \; j = 0, 1, 2, ... \end{aligned} \qquad (3.7)$$

Finally, the solution λ_j can be approximated by the truncated series

$$\phi_k = \sum_{j=0}^{k-1} \lambda_j \qquad (3.8)$$

that

$$\lim_{k \to \infty} \phi_k = \lambda \qquad (3.9)$$

In computing λ_n, choosing large values for n, increasing the number of terms in the expression of A_j and this causes propagation of round off errors. The ADM reduces significantly the massive computation which may arise if discretization methods are used. The convergence series was investigated by several authors [12, 11].

4 NUMERICAL EXAMPLES

In this section we shall illustrate the technique by different matrices. Dealing with matrices of known eigenvalues allow for more error analysis, in the first example we consider a 3×3 matrix in which we examine the accuracy and validity of our algorithm

Example 4.1. *Consider the matrix*

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

The matrix A has distinct eigenvalues $1, 2, 3$, where the characteristic equation is given by $p_A(\lambda) = \lambda^3 - 12\lambda^2 + 29\lambda - 18 = 0$, or, in a fixed-point form, $\lambda = \frac{-1}{29}\lambda^3 + \frac{12}{29}\lambda^2 + \frac{18}{29}$. According to what we discussed, we set

$$\lambda_0 = \frac{18}{29}, \quad \lambda_{j+1} = \frac{-1}{29}A_j + \frac{12}{29}B_j, \quad j = 0, 1, ...$$

where A_j, B_j are the Adomian polynomials for λ^3, λ^2 respectively. The first 10 decomposition components are listed as: with $\lambda_0 = 0.62069$, $\lambda_1 = \frac{-1}{29}A_0 + \frac{12}{29}B_0$, where the Adomian polynomials $A_0 = \lambda_0^3$ and $B_0 = \lambda_0^2$

$\lambda_2 = 0.0716276$, $\lambda_3 = 0.0419275$, $\lambda_4 = 0.0273175$, $\lambda_5 = 0.0189993$, $\lambda_6 = 0.0189993$, $\lambda_7 = 0.0138099$, $\lambda_8 = 0.0128217$, $\lambda_9 = 0.0091533$, $\lambda_{10} = 0.00716222$. Approximately we have $\lambda = \sum_{i=0}^{10} \lambda_i = 0.993678$. The accurate value of this eigenvalue is 1. Of course the accuracy can be improved by computing more terms of the approximate solution by taking more terms of the series in (3.3).

Example 4.2. *Consider the Toplitz matrix $I^{(2)}$ of size 4,*

$$I_4^{(2)} = \begin{pmatrix} \frac{-\pi^2}{3} & 2 & \frac{-1}{2} & \frac{2}{9} \\ 2 & \frac{-\pi^2}{3} & 2 & \frac{-1}{2} \\ \frac{-1}{2} & 2 & \frac{-\pi^2}{3} & 2 \\ \frac{2}{9} & \frac{-1}{2} & 2 & \frac{-\pi^2}{3} \end{pmatrix}$$

Using Mathematica, it is easy to verify that the smallest eigenvalue is $|-0.4351| = 0.4351$, and the characteristic equation for finding the eigenvalues is given by

$$p_{I^2}(\lambda) = \lambda^4 + 13.1595\lambda^3 + 52.39\lambda^2 + 68.7453\lambda + 21.043.$$

The characteristic equation can be written in fixed-point form as

$$\lambda = \frac{-21.043}{68.7453} - \frac{52.39}{13.1595}\lambda^2 - \frac{13.1595}{68.7453}\lambda^3 - \frac{1}{68.7453}\lambda^4$$

According to equation (3.7), we set

$$\lambda_0 = \frac{-21.043}{68.7453}, \quad \lambda_{j+1} = -\frac{52.39}{13.1595}A_{2j} - \frac{13.1595}{68.7453}A_{3j} - \frac{1}{68.7453}A_{4j}, \ j = 0, 1, 2, ... \tag{4.1}$$

where A_{2j}, A_{3j}, A_{4j} are the Adomian's polynomials for $\lambda^2, \lambda^3, \lambda^4$ respectively. The first few Adomian's polynomials are given by

$$A_{20} = \lambda_0^2, \quad A_{21} = 2\lambda_0\lambda_1, \quad A_{22} = 2\lambda_0\lambda_2 + \lambda_1^2, ...$$

$$A_{30} = \lambda_0^3, \quad A_{31} = 3\lambda_0^2\lambda_1, \quad A_{32} = 3\lambda_0^2\lambda_2 + 3\lambda_0\lambda_1^2, ...$$

and,

$$A_{40} = \lambda_0^4, \quad A_{41} = 4\lambda_0^3\lambda_1, \quad A_{42} = 6\lambda_0^2\lambda_1^2 + 4\lambda_0^3\lambda_2, ...$$

From equation (4.1), the first three iterations are given by

$$\lambda_1 = -\frac{52.39}{13.1595}A_{20} - \frac{13.1595}{68.7453}A_{30} - \frac{1}{68.7453}A_{40} = -0.0660435$$

$$\lambda_2 = -\frac{52.39}{13.1595}A_{21} - \frac{13.1595}{68.7453}A_{31} - \frac{1}{68.7453}A_{41} = -0.0273693$$

$$\lambda_3 = -\frac{52.39}{13.1595}A_{22} - \frac{13.1595}{68.7453}A_{32} - \frac{1}{68.7453}A_{42} = -0.0139351$$

and so for the remaining components. Consequently, the series solution

$$\phi_6 = \phi_k = \sum_{j=0}^{5} \lambda_j = -0.433766$$

The absolute error relative to the exact solution is

$$Error = |Exact - \phi_6| = 1.33 \times 10^{-3}$$

To find the second eigenvalue, we treat a new characteristic function given by

$$\frac{\lambda^4 + 13.1595\lambda^3 + 52.39\lambda^2 + 68.7453\lambda + 21.043}{\lambda + 0.433766}$$
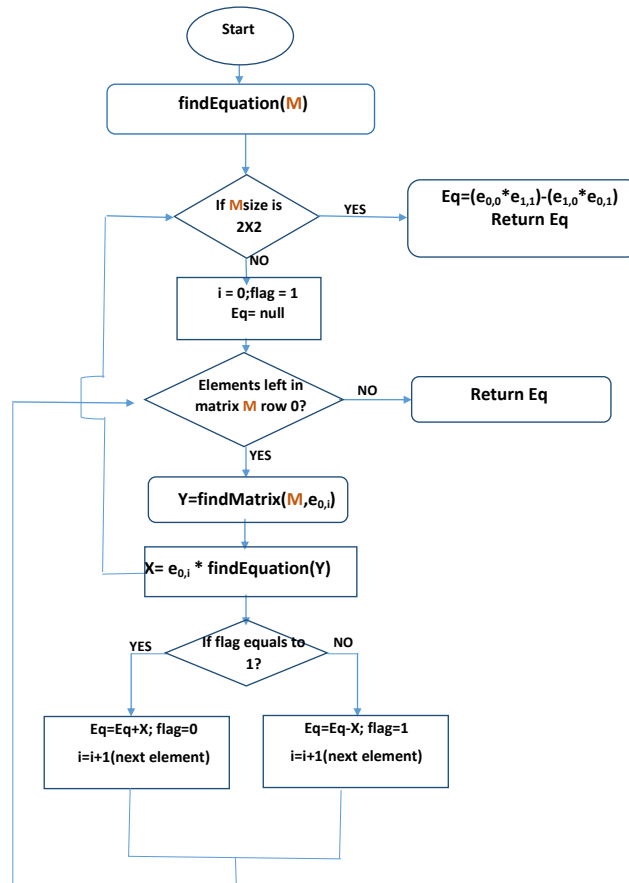
then we repeat the same procedure as in above.

Fig. 1. General Flow Chart to find the characteristic polynomial

5   CONCLUDING REMARKS

We have shown that a novel algorithm based on Adomian decomposition method is proposed to obtain eigenvalues of some Toeplitz matrices. The algorithm is simple and computationally durable. It is concluded that the ADM is reliable and efficient.

findEquation(M)

If the size of matrix M is 2 X 2
    Equation eq= (element $_{0,0}$*element $_{1,1}$)-(element$_{0,1}$* element $_{1,1}$)
Otherwise
    Repeat for each elementin the first row (for each element e $_{0,i}$repeat)
        {
        Y=findMatrix(M,e $_{0,i}$)
        Equation eqi = e$_{0,i}$* findEquation(Y)
        If(i % 2 ≠ 0)
            Equation eq = eq + eqi
        Etherwise
            Equation eq = eq - eqi
        }

findMatrix (M, e $_{0,i}$)
Y=Copy of all elements of M excluding elements in the first row and elements in column i

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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