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On the Stability Analysis of MDGKN Systems with Control Parameters

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Authors' contributions

This work was carried out in collaboration between both authors. Author UDA designed the study, performed the analysis, wrote the protocol and wrote the first draft of the manuscript. Author MOO proof read the manuscript and effected the corrections. Both authors read and approved the final manuscript.

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Abstract

In this work, we study linear systems with Mass, Damping Force, Gyroscopic Force, Stiffness and Circulatory Force (MDGKN systems) with control parameters. The relationship between the parameters determines the stability or otherwise of the system. The Lyapunov direct method is used to analyse MDGKN system. Stability theorem for determining the stability or otherwise of MDGKN is formulated. The results are illustrated on a 2x2 and a 3x3 matrix systems to show the effectiveness of the results obtained.

Keywords: Linear systems; control parameters; Lyapunov method; stability.

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1 Introduction

Linear systems involving Mass, Damping Force, Gyroscopic Force, Stiffness and Circulatory Force describe the damped gyroscopic system with circulatory effect known as the MDGKN systems. These systems arise in the modeling of mechanical systems with follower forces, in formulation of rotor systems with internal damping and with sliding bearings, in turbines with unsymmetrical steam flow, in articulated pipes, etc [1]. For almost a century it has been well known that circulatory forces can cause instabilities [2]. In many engineering and physical applications, it is vital to know how the stability is improved or destroyed when these forces are taken into account [3-9]. In this work, we study the stability properties of the MDGKN systems and formulate stability theorem for determining the stability or otherwise of MDGKN systems with control parameters. Examples are given to demonstrate the results obtained.

2 Methodology

Consider the non-conservative linear system of the form

$$M\ddot{x} + (\delta D + \mu G)\dot{x} + (K + \nu N)x = f(t) \tag{1}$$

where the dot denotes the time derivative; $x \in R^n$; and

$D = D^* > 0, G = -G^* > 0 \; and \; N = -N^*$ are real matrices corresponding to dissipative, gyroscopic and circulatory forces. The magnitudes are controlled by the parameters δ, μ and ν, respectively while $f(t)$ describes excitation. The relationship between the control parameters determines the stability or instability status of system (1). We now examine the following cases:

Case 1: $\delta \sim \nu \ll \mu$

The most interesting in practice is the situation when these forces in the system are small as compared with the gyroscopic force. The critical gyroscopic parameter μ on the boundary of the gyroscopic stabilization domain of the non-conservative system is a function of the parameters corresponding to the dissipative and circulatory forces. Moreover, stability is extremely sensitive to the choice of a perturbation while the balance of forces leading to the asymptotic stability is not obvious.

Case 2: $\mu \ll \delta \sim \nu$

This case with a tendency for high perturbation arising from very small gyroscopic effect may be unstable and will not be considered since the bounds of solutions are not obtained for unstable systems.

In the following we assume the situation of case 1 where the stability is ensured and for simplicity we omit the parameters and proceed with the analysis.

If excitation in eqn (1) is negligible $i.e \; f(t) = 0$, we obtain the following homogeneous linear system

$$M\ddot{x} + (D + G)\dot{x} + (K + N)x = 0 \tag{2}$$

The stability or otherwise of system (2) can be determined by eigenvalue method and also by Routh-Hurwitz method [10]. In this work, we shall use Lyapunov direct method to analyse the stability or otherwise of the system. The advantage of this method over eigenvalue method is that the stability status of the system can be determined easily in cases where the eigenvalues cannot be found easily. Applying the direct method of Lyapunov, we have the following:

The system (2) is equivalent to the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -M^{-1}(D+G)x_2 - M^{-1}(K+N)x_1 \tag{3}$$

Putting (3) in the equivalent form of a first order system we have the following

$$\dot{z} = Az \tag{4}$$

where $A = \begin{bmatrix} O & I \\ -M^{-1}(K+N) & -M^{-1}(D+G) \end{bmatrix}$ (I is the identity matrix and O is the zero matrix).

We define a function *V(z(t))* called a Lyapunov function for system (4) if *V > 0* and the time derivative $\dot{V} \leq 0$ for all solutions *z(t)* of (4). The existence of such a Lyapunov function implies stability of the system (asymptotic stability if $\dot{V} < 0$) [11,12]. Let

$$V = z(t)^* Pz(t)$$

be the Lyapunov function with a Hermitian matrix *P>0.* For the solutions of (4) we then have $V = z(t)^*(A^*P + PA)z(t)$, such that condition $\dot{V} \leq 0$ is expressed by the matrix $Q = Q^* \geq 0$ of the Lyapunov matrix equation.

$$A^*P + PA = -Q$$

The system (4) (and therefore also system (2) is asymptotically stable, if there exist Hermitian matrices *P>0* and *Q>0* which satisfy the Lyapunov matrix equation.

2.1 Derivation of P and Q

To derive suitable positive definite Hermitian matrices P and Q from the Lyapunov function of the dynamical system (2), we start with the energy equation which is a first integral of the equations of motion. By multiplying (2) from the left with $\dot{x}^*(t)$ and adding the complex transpose of this equation we get

$$\dot{x}^*M\dot{x} + x^*Kx + 2\int_0^t x^*D\dot{x}dt + \int_0^t(\dot{x}^*Nx - x^*N\dot{x})dt = 2E_0 \tag{5}$$

where $E_0 = {}^1\!/_2\,(\dot{x}^*(0)M\dot{x}\,(0) + x^*(0)Kx(0))$ is the initial mechanical energy of the system. It is obvious that we cannot use the energy $V = \dot{x}^*M\dot{x} + x^*Kx > 0$ as a Lyapunov function since the sign of $\dot{V}$ is indefinite due to the circulatory forces described by $N \neq 0$. The idea is now to add terms to (5) to obtain a function V for which $\dot{V}$ is negative definite. For this purpose, we construct a new first integral of the system (2) by now multiplying from left with $\dot{x}(t)$ and adding the complex transpose of this new equation. This leads to

$$(\dot{x}^*Mx + x^*M\dot{x} + x^*Dx) + \int_0^t(2x^*K\dot{x} - 2\dot{x}^*M\dot{x} + (x^*G\dot{x} - \dot{x}^*Gx))dt = c \tag{6}$$

where c is an integration constant. To find a Lyapunov function we introduce a proper positive constant γ, which has to be determined. Multiplying equation (6) by ${}^{\gamma}\!/_2$ and adding (5) and (6) we get after rearranging terms the following:

$$x^*\left(K+\frac{\gamma}{2}D\right)x+\dot{x}^*M\dot{x}+\frac{\gamma}{2}(\dot{x}^*Mx+x^*M\dot{x})=2E_0+\frac{\gamma}{2}c$$
$$-\int_0^t(\gamma x^*Kx+\tfrac{\gamma}{2}(x^*G\dot{x}-\dot{x}^*Gx)+(\dot{x}^*Nx-x^*N\dot{x})+\dot{x}^*(2D-\gamma M)\dot{x})dt \qquad (7)$$

putting (7) in the quadratic form, $V=z^*Pz$

where $z^*=\begin{bmatrix}x^*\\ \dot{x}^*\end{bmatrix}$ and $z=\begin{bmatrix}x\\ \dot{x}\end{bmatrix}$

we have that

$$V=\begin{bmatrix}x^*\\ \dot{x}^*\end{bmatrix}\begin{bmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{bmatrix}[x\ \ \dot{x}] \qquad , \qquad V=\begin{bmatrix}x^*P_{11}x & x^*P_{12}\dot{x}\\ \dot{x}^*P_{21}x & \dot{x}^*P_{22}\dot{x}\end{bmatrix}$$

But eqn (7) is in the form

$$V=P+\int_0^t Q(s)ds$$

Where using (7) we have

$$P=\begin{bmatrix}K & \frac{\gamma}{2}M\\ \frac{\gamma}{2}M & M\end{bmatrix}, \qquad Q=\begin{bmatrix}\gamma K & \frac{\gamma}{2}G\\ \frac{\gamma}{2}G* & 2G-\gamma M\end{bmatrix} \qquad (8)$$

where γ is a real number.

3 Stability Analysis

Considering the MDGKN system with control parameters, we proceed with the analysis as follows: We assume that $M=M^*>0,\ \ D=D^*>0\ \ and\ K=K^*>0$. Since P and Q must be positive definite, we are interested in finding the condition for the existence of a real number γ that guarantees the positive definiteness of P and Q. Schur's lemma provides this condition.

3.1 Schur's lemma

$A\ matrix\ R\begin{bmatrix}R_1 & R_2\\ R_2^* & R_3\end{bmatrix}\ \ with\ Hermitian\ submatrices\ R_1\ and\ R_3\ is\ positive\ definite$
$if\ and\ only\ if\ R_1\ and\ R_3-R_2^*R_1^{-1}R_2\ are\ positive\ definite$[11].

Applying the lemma to Q given by (8), we get that Q>0 if and only if there exists γ >0 such that

$$2D-\gamma M-\left(-N+\tfrac{\gamma}{2}G\right)^*(\gamma K)^{-1}\left(-N+\tfrac{\gamma}{2}G\right)>0$$

Rearranging terms we get the following conditions

$$-\gamma^2\left(M+\tfrac{1}{4}G^*K^{-1}G\right)+\gamma\left(2D+\tfrac{1}{2}(G^*K^{-1}N+N^*K^{-1}G)\right)-N^*K^{-1}N>0 \qquad (9)$$

We define all $z\in C^n$, then (9) is equivalent to the inequality

$$-\gamma^2 z^*\left(M+\tfrac{1}{4}G^*K^{-1}G\right)z+\gamma z^*\left(2D+\tfrac{1}{2}(G^*K^{-1}N+N^*K^{-1}G)\right)z-z^*N^*K^{-1}Nz>0 \tag{10}$$

Taking $z^*z=I$, the coefficients of the quadratic polynomial in γ are Rayleigh quotients for Hermitian matrices. These Rayleigh quotients are limited by the smallest eigenvalue λ_{min} and the largest eigenvalue λ_{max} of the respective matrices [11,13]. The Rayleigh quotients for the matrices $M, D, G^*K^{-1}G, M+\frac{1}{4}G^*K^{-1}G\ and\ N^*K^{-1}N$ are all positive since $M, D, and\ K^{-1}$ are assumed to be positive definite. Introducing the scalars $a, b\ and\ c$ defined by

$$\left.\begin{array}{c} a=\lambda_{max}\left(M+\frac{1}{4}G^*K^{-1}G\right)>0 \\ b=\lambda_{min}(2D+\frac{1}{2}(G^*K^{-1}N+N^*K^{-1}G) \\ c=\lambda_{max}(N^*K^{-1}N)>0 \end{array}\right\} \tag{11}$$

Inequality (10) is now satisfied if there exists γ>0 with

$$-\gamma^2 a+\gamma b-c>0 \tag{12}$$

There are solutions if and only if

$$b^2-4ac>0 \quad and\ \ b>0 \tag{13}$$

In this case γ can be chosen as any number in the interval

$$\frac{b-\sqrt{b^2-4ac}}{2a}<\gamma<\frac{b+\sqrt{b^2-4ac}}{2a} \tag{14}$$

and then matrix Q will be positive definite.

Next, if $Q>0$ then $2D-\gamma M>0$. This implies $\frac{\gamma}{2}D-\frac{\gamma^2}{4}M>0$.

Also, $P>0$ and applying the Schur's lemma on P we have

$$M-\frac{\gamma}{2}M(K+\frac{\gamma}{2}D)^{-1}\frac{\gamma}{2}M>0$$

Multiplying through by $(K+\frac{\gamma}{2}D)$ we have

$$M\left(K+\frac{\gamma}{2}D\right)-\frac{\gamma}{2}M\left(K+\frac{\gamma}{2}D\right)^{-1}\left(K+\frac{\gamma}{2}D\right)\frac{\gamma}{2}M>0$$
$$M\left(K+\frac{\gamma}{2}D-\frac{\gamma^2}{4}M\right)>0$$
$$\Rightarrow K+\frac{\gamma}{2}D-\frac{\gamma^2}{4}M>0$$

We now formulate the following theorem that provides the condition for the stability of system (1) and then asymptotic stability of system (2).

3.2 Stability theorem

$Assume\ a, b\ and\ c\ defined\ by\ (11).\ If\ b^2 - 4ac > 0\ and\ b > 0,$
$then\ system\ (1)\ is\ asymptotically\ stable.$

To apply the theorem, it will be beneficial and less cumbersome to estimate a and b as follows

$$a \leq \lambda_{max}(M) + \frac{1}{4}\lambda_{max}(G^*K^{-1}G)$$
$$b \geq 2\lambda_{min}(D) + \frac{1}{2}\lambda_{min}(G^*K^{-1}N + N^*K^{-1}G) \tag{15}$$

Thus,

$$\lambda_{max}(G^*K^{-1}G) \leq g_{max}^2/k_{min}\ ,$$
$$\lambda_{max}(N^*K^{-1}N) \leq n_{max}^2/k_{min} \tag{16}$$
$$\lambda_{min}(G^*K^{-1}N + N^*K^{-1}G) \geq -2g_{max}n_{max}/k_{min}\ ,$$

where $g_{max} = |\lambda(G)|_{max}$, $n_{max} = |\lambda(N)|_{max}$ are the maximum of the absolute values of the eigenvalues of G and N, respectively, and k_{min} is the smallest eigenvalue of $K > 0$ [14]. Additionally we use m_{max} for $\lambda_{max}(M)$ and d_{min} for $\lambda_{min}(D)$. Applying (15) and (16) conditions (13) for the existence of $\gamma > 0$ become

$$\left(2d_{min} - \frac{g_{max}n_{max}}{k_{min}}\right)^2 - \frac{4\left(m_{max} + \frac{\frac{1}{4}g_{max}^2}{k_{min}}\right)n_{max}^2}{k_{min}} > 0\ , \tag{17}$$
$$2d_{min} - \frac{g_{max}n_{max}}{k_{min}} > 0$$

Obviously, both inequalities in (17) are satisfied if

$$d_{min}^2k_{min} - d_{min}g_{max}n_{max} - m_{max}n_{max}^2 > 0 \tag{18}$$

(18) is a more restrictive condition than (13) and contains the smallest and largest eigenvalues of the system matrices. It is therefore a simple sufficient condition for asymptotic stability of system (2). Choosing appropriate $\gamma > 0$ by adding the two limits in (14) we have $\gamma = \frac{b}{a}$

And using the estimates of a and b we have the following

$$\gamma = (d_{min}k_{min} - \tfrac{1}{2}g_{max}n_{\max})/(m_{max}k_{min} + \tfrac{1}{4}g_{max}^2) \tag{19}$$

This condition is sufficient for asymptotic stability of (2) [15,16,11,17,18].

4 Applications

Example 1

To illustrate the formulas for response bounds for the inhomogeneous system let us consider the 2x2 system described by

$$\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \left(\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}\right)\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \left(\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = f(t) \tag{20}$$

We obtain the constants $a, b\ and\ c$ defined in (11) as follows:

$$a = \lambda_{max}\left(M + \frac{1}{4}G^*K^{-1}G\right) = 7$$
$$b = \lambda_{min}(2D + \frac{1}{2}(G^*K^{-1}N + N^*K^{-1}G) = \frac{32}{5}$$
$$c = \lambda_{max}(N^*K^{-1}N) = 1$$

Applying the values on the stability condition we have that

$$b^2 - 4ac = 12.96 > 0$$

The system is therefore stable according to the stability theorem.

Example 2

Consider the 3x3 system

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \left(\begin{bmatrix} 8 & -2 & 2 \\ -2 & 8 & -2 \\ 2 & -2 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}\right)\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} + \left(\begin{bmatrix} 4 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (21)$$

We compute the constants defined in (11) as follows:

$$a = \lambda_{max}\left(M + \frac{1}{4}G^*K^{-1}G\right) = \frac{37+\sqrt{809}}{8}$$
$$b = \lambda_{min}(2D + \frac{1}{2}(G^*K^{-1}N + N^*K^{-1}G) = 13$$
$$c = \lambda_{max}(N^*K^{-1}N) = 3$$

Since $b > 0\ and\ b^2 - 4ac = 70.8 > 0$, the system is stable according to the stability theorem.

5 Conclusion

The gyroscopic systems (damped and undamped) are generally stable systems but the addition of circulatory forces can destroy stability. For the MDGKN systems with control parameters, the relationship between the control parameters determines the stability or otherwise of the systems. Stability theorem for determining the stability or otherwise of MDGKN is formulated. The results are illustrated on a 2x2 and a 3x3 matrix systems to show the effectiveness of the obtained results. With these results, the stability status of MDGKN systems can be determined without explicit computation of eigenvalues or in situations where the eigenvalues cannot be computed easily.

Competing Interests

Authors have declared that no competing interests exist.

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