On Propositions Pertaining to the Riemann Hypothesis III

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Short Research Article

Abstract

In this paper, we consider further propositions concerning the range of possible distributions over the unit circle, for the Riemann zeta function as in prior research. We also derive some new upper bounds on the sum of norms for the tail sequence corresponding to the Riemann zeta function. We discuss some hypotheses, conditional on which, properties of concentrated distributions may be obtained. Specific sub-classes are shown of distributions that occur infinitely often along the imaginary axis.

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1 Introduction

The subject of study for this paper is the Riemann zeta function and the Riemann Hypothesis ([1], [2]). The point of view is in line with and extends prior research in [3],[4] and [5]. The main interest is in understanding the pattern of zeros and non-zeroes. We draw connections with associated distributions for complex functions defined by an absolutely convergent series and consider concentration phenomena in associated probability measures over the unit circle, for the given series. Such an undertaking is linked to solving for bounds on the tail of the sequence that defines the series. This corresponds to the quantity $Z^*(\sigma, t, m)$ in this paper, which is the sum of norms for the series, the tail from m onwards, with complex input (σ, t). Our interest lies in considering upper bounds on $Z^*(\sigma, t, m)$. We also study an optimization problem with non-linear constraints, which solves for the minimum value $t^*(m, r)$ needed for the construction of points $(\sigma, t^*(m, r))$ which concentrate the measure on an arc. Here, m denotes the number of points to be positioned on the arc and $0 < r < 1$ is a level of accuracy associated with positioning. By making certain hypotheses on $Z^*(\sigma, t, m)$ and $t^*(m, r)$, we are allowed an analysis of regions where the associated distributions are concentrated, when the input complex variable is in the critical region. To further highlight this phenomenon, and other related phenomena, we also study the prime zeta function, where we analyse the existence and occurence of points that correspond to equitable distributions i.e. distributions which are not concentrated. Previous research on this subject may be found in [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16] and [17].

2 Probability Measures Over the Unit Circle

We present the framework for the analysis in this paper. For the appropriate notation, we refer to [3], [4] and [5]. The concentration theorem is as follows.

Proposition 2.1. *([4]) Let μ be a probability measure on the unit circle $\mathbb{S}^1$. Then,*

$$\mathbb{E}_\mu[z] \neq 0 \text{ if there exist numbers } 0 \leq \theta' \leq \theta'' \leq 2\pi \text{ such that } \theta'' - \theta' \leq \pi \text{ and}$$

$$\mu(\{z : \theta' \leq \theta(z) \leq \theta''\}) > \frac{1}{1 + \cos\left(\frac{\theta'' - \theta'}{2}\right)}. \tag{2.1}$$

Of course, by symmetry of the circle, this also applies to arcs of the form described by $0 \leq \theta' \leq \theta'' \leq 2\pi$ and $\theta' + (2\pi - \theta'') \leq \pi$, requiring the concentration condition

$$\mu(\{z : \theta(z) \in [0, \theta'] \cup [\theta'', 2\pi]\}) > \frac{1}{1 + \cos\left(\frac{\theta' + (2\pi - \theta'')}{2}\right)}. \tag{2.2}$$

For each complex variable ([2], [18]) given by $(\sigma, t) \in S = \{(\sigma, t) \in \mathbb{R}^2 : \sigma \in (0, 1); t \neq 0\}$, we define $Z_0(s) = (1, 0)$; $Z_n(s) = \frac{1}{(2n+1)^s} - \frac{1}{(2n)^s}$, for each $n \in \mathbb{Z}^+$. Hence, the Riemann zeta function is $\zeta(s) = (\frac{1}{1-2^{1-s}}) \times \sum_{n \geq 0} Z_n(s)$. We also define $\zeta^*(s) = \sum_{n \geq 0} Z_n(s)$. For $s = (\sigma, t) \in S$, the probability measure μ_s on $\mathbb{S}^1$ is defined as follows. For each measurable subset $A \subseteq \mathbb{S}^1$,

$$\mu_s(A) = \frac{\sum_{n \geq 0: Z_n(s) \in A} ||Z_n(s)||}{\sum_{n \geq 0} ||Z_n(s)||}. \tag{2.3}$$

Consider the following definition.

Definition 2.1. *A distribution μ_s is said to be concentrated if either condition 2.1 or 2.2 is satisfied. The set of concentrated points in S is defined as $S^* = \{s \in S : \mu_s \text{ is concentrated}\}$.*

We are interested in providing an upper bound for the sum of norms corresponding to the tail i.e.

$$Z^*(\sigma, t, m) := \sum_{n \geq m+1} ||Z_n(s)||. \tag{2.4}$$

Before proceeding, we introduce some notation. Suppose we are given two real-valued functions defined on a subset of $X \subseteq \mathbb{R}^d$ i.e $f, g : X \to \mathbb{R}$. We say that $f(x) = O(g(x))$ if for some $\phi > 0$ and $y > 0$, $|f(x)| \leq \phi |g(x)|$ for each $x \in X$ such that $\max_{i \in \{1, \dots d\}} x_i \geq y$. This concerns the asymptotic order or the rate of growth of f by an upper bound that is provided by the function g.

Consider the following hypothesis. See also [19].

Hypothesis 2.1. *For each $\sigma \in [0.5, 1)$, there exists $\alpha \in (0, 1)$ and $k \in \mathbb{N}$ such that $Z^*(\sigma, t, m) = O((\ln(|t|))^k m^{-\alpha})$.*

The purpose of considering a bound on the tail is to be able to construct t values i.e. imaginary parts of the complex input, which position the points in the set $\{Z_n(s)\}_{1 \leq n \leq m}$ such that the points $Z_n(s)$ are all contained in an arc, which corresponds to a subset of $[0, 2\pi]$. For large m, the sum of norms $Z^*(\sigma, t, m)$ would be small, resulting in the distribution $\mu_{(\sigma, t)}$ to be concentrated on the arc. Since m is large, the t values needed for this construction would be large, which would increase the value of a bound on $Z^*(\sigma, t, m)$ that is increasing in $|t|$. For example, consider the bound from [4] which leads to $Z^*(\sigma, t, m) \leq \left(1 + \frac{|t|}{\sigma}\right) \frac{1}{(2m+2)^\sigma}$. We now show some different bounds on $Z^*(\sigma, t, m)$. Note that

$$
\begin{aligned}
Z_n(s) &= \frac{1}{(2n+1)^s} - \frac{1}{(2n)^s} \\
&= \left(\frac{1}{(2n)^s}\right) \times \left(\left(\frac{2n}{2n+1}\right)^s - 1\right).
\end{aligned}
\tag{2.5}
$$

Hence,

$$
\begin{aligned}
||Z_n(s)|| &= \left\lVert \frac{1}{(2n)^s} \right\rVert \times \left\lVert \left(\frac{2n}{2n+1}\right)^s - 1 \right\rVert \\
&= \left(\frac{1}{(2n)^\sigma}\right) \times \left\lVert \left(\frac{2n}{2n+1}\right)^\sigma \left(\cos\left(t\ln\left(\frac{2n}{2n+1}\right)\right), \sin\left(t\ln\left(\frac{2n}{2n+1}\right)\right)\right) - (1, 0) \right\rVert \\
&= \sqrt{((2n)^{-\sigma} - (2n+1)^{-\sigma})^2 + 2(2n)^{-\sigma}(2n+1)^{-\sigma}\left(1 - \cos\left(|t|\ln\left(1 + \frac{1}{2n}\right)\right)\right)} \\
&\leq (2n)^{-\sigma} - (2n+1)^{-\sigma} + 2(2n)^{-\sigma}\left|\sin\left(0.5|t|\ln\left(1 + \frac{1}{2n}\right)\right)\right|
\end{aligned}
\tag{2.6}
$$

The last inequality follows from the fact that for any two real numbers $x, y \geq 0$, it follows that $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ and also from the equality $\sqrt{2}|\sin(0.5x)| = \sqrt{1 - \cos(x)}$. Further,

$$
\begin{aligned}
(2n)^{-\sigma} - (2n+1)^{-\sigma} + 2(2n)^{-\sigma}\left|\sin\left(0.5|t|\ln\left(1 + \frac{1}{2n}\right)\right)\right| &\leq (2n)^{-\sigma} - (2n+1)^{-\sigma} + 4(2n)^{-\sigma}\ln\left(1 + 0.5|t|\ln\left(1 + \frac{1}{2n}\right)\right) \\
&\leq (2n)^{-\sigma} - (2n+1)^{-\sigma} + 4(2n)^{-\sigma}\ln\left(1 + \frac{|t|}{4n}\right).
\end{aligned}
$$

In the first inequality above, we apply the fact that for any $x \geq 0$, we have that $|\sin(x)| \leq 2\ln(1+x)$. In the second inequality, we apply the fact that for any $x \geq 0$, we have $\ln(1+x) \leq x$, and that $\ln(x)$ is an increasing function. Hence, it follows that

$$
\sum_{n \geq m+1} ||Z_n(s)|| \leq \sum_{n \geq m+1} (2n)^{-\sigma} - (2n+1)^{-\sigma} + 4(2n)^{-\sigma}\ln\left(1 + \frac{|t|}{4n}\right).
$$

Since $\ln(x)$ is increasing in x and n^{-1} is decreasing in n, the latter term, which depends on t, in the above sum

may be bounded above by a definite integral as

$$\sum_{n \geq m+1} 4(2n)^{-\sigma} \ln\left(1 + \frac{|t|}{4n}\right) \quad \leq \quad 4 \int_{m+1}^{\infty} x^{-\sigma} \ln(1 + x^{-1}|t|)dx \tag{2.7}$$

$$= \quad 4 \int_{\left(\frac{m}{m+1}, 1\right)} (1 - \omega')^{\sigma-2} \ln(1 + (1 - \omega')|t|)d\nu(\omega') \tag{2.8}$$

$$= \quad 4 \int_{\left(0, \frac{1}{m+1}\right)} \omega^{\sigma-2} \ln(1 + \omega|t|)d\nu(\omega), \tag{2.9}$$

where ν is the uniform probability measure on $(0, 1)$. The above holds due to the change of variables substitution $(1 - \omega')^{-1} = x$. Another change of variables $\omega = 1 - \omega'$ is applied, however, the last equality holds due to the symmetry of the density of the uniform probability measure around 0.5. We obtain the expectation of an unbounded random variable in ω over the measurable set $(0, \frac{1}{m+1})$.

One may also derive the polar form of $-Z_n(s)$. This is useful since, by varying t, one would position $-Z_n(s)$ in a given arc where the distribution is to be concentrated. Note that if $Z_n(s)$ leads to concentration on an arc, then so does $-Z_n(s)$ on the arc that is polar opposite on the unit circle.

Since $-Z_n(s) = \left(\frac{1}{(2n)^s}\right) \times \left(1 - \left(\frac{2n}{2n+1}\right)^s\right)$, one need only derive the angle of $1 - \left(\frac{2n}{2n+1}\right)^s$, then add it to the angle of $\frac{1}{(2n)^s}$ which is $-t\ln(2n)$. Note that as $\left(\frac{2n}{2n+1}\right)^s = \left(\frac{2n}{2n+1}\right)^\sigma \left(\cos\left(t\ln\left(\frac{2n}{2n+1}\right)\right), \sin\left(t\ln\left(\frac{2n}{2n+1}\right)\right)\right)$, by geometry, we may show that the angle $\hat{\theta}_n(\sigma, t)$ for $1 - \left(\frac{2n}{2n+1}\right)^s$ is

$$\hat{\theta}_n(\sigma, t) = \tan^{-1}\left(\frac{\sin(t\ln(\frac{2n}{2n+1}))}{\left(\frac{2n+1}{2n}\right)^\sigma - \cos(t\ln(\frac{2n}{2n+1}))}\right). \tag{2.10}$$

Hence, the polar form of $-Z_n(s)$ may be derived as

$$-Z_n(s) = ||Z_n(s)||(\cos(-t\ln(2n) - \hat{\theta}_n(\sigma, t)), \sin(-t\ln(2n) - \hat{\theta}_n(\sigma, t))). \tag{2.11}$$

Now, we shall define an optimization problem based on the above polar form.

$$\min_{(t, (q_n)_{n=1}^m) \in \mathbb{R} \times \mathbb{Z}^{m-1}} t$$

$$\text{subject to}: \left| -t\left(\frac{\ln(2n)}{2\pi}\right) - \frac{\theta_n(\sigma, t)}{2\pi} - q_n \right| \leq \frac{r}{2\pi} \text{ for all } 1 \leq n \leq m. \tag{2.12}$$

$$2(2n)^{-\sigma}\left| \sin\left(0.5t\ln\left(1 + \frac{1}{2n}\right)\right) \right| \leq \frac{\sqrt{2} - 1}{\sqrt{2}m} \text{ for all } 1 \leq n \leq m. \tag{2.13}$$

$$q_n \leq 0 \text{ for all } 1 \leq n \leq m. \tag{2.14}$$

$$\sum_{n=1}^m q_n \leq -1. \tag{2.15}$$

$$t \geq 0. \tag{2.16}$$

Note that the optimization problem has non-linear constraints ([20],[21],[22]). Denote as $t^*(m, r)$, the optimal value (minimum) of the above optimization problem (minimization, for a given $m \in \mathbb{N}$ and $0 < r < 1$. The optimal value (minimum) is finite i.e. an optimizer (minimizer) exists, since the set of feasible t values is non-empty and closed in $\mathbb{R}_+$. Non-emptiness would follow from simultaneous Diophantine approximation ([23],[24]) of the real numbers $(\frac{-\ln(2n)}{2\pi}, \ln(\frac{2n}{2n+1}))_{1 \leq n \leq m}$, applying continuity of $\theta_n(\sigma, t)$ in t. The continuity also yields closedness of the feasible t region. Note also that $t^*(m, r)$ would be non-decreasing in m, given that more constraints are added with larger m.

Consider the following hypotheses

Hypothesis 2.2. *For each $r \in (0,1)$, $\lim_{m \to +\infty} t^*(m,r) = +\infty$.*

Hypothesis 2.3. *For each $r \in (0,1)$, there exists $Q \in \mathbb{N}$ such that $t^*(m,r) = O(m^Q)$.*

Now, we prove the following proposition.

Proposition 2.2. *Suppose that Hypothesis 2.1, Hypothesis 2.2 and Hypothesis 2.3 are satisfied. Suppose that $\sigma \in [0.5, 1)$. Then, there exist countably many pairwise disjoint intervals $\{[\underline{t}_k, \overline{t}_k]\}_{k \in \mathbb{Z}^+}$, such that $\lim_{k \to \infty} \underline{t}_k \to +\infty$ and for each k and $t \in [\underline{t}_k, \overline{t}_k]$, we have that $(\sigma, t) \in S_+^*$ i.e. $\mu_{(\sigma,t)}$ is concentrated.*

Proof. Consider the following function of m,

$$Z^*(\sigma, t^*(m,r), m). \tag{2.17}$$

Note that by the given hypothesis, we have that $Z^*(\sigma, t^*(m,r), m) = O((Q \ln(m))^k m^{-\alpha})$, which converges to zero as $m \to +\infty$. Hence, for small r, by choosing $0 \leq t \leq t^*(m,r)$, we may simultaneously diminish the norm of the tail $Z^*(\sigma, t, m)$ and also position the points $\{Z_n(s)\}_{1 \leq n \leq m}$ inside an arc around $(-1, 0)$, the sum of norms over $1 \leq n \leq m$ upper bounded strictly less than 1 and the probability measure $\mu_{(\sigma,t)}$ would be concentrated on an arc around $(1,0)$. $\qquad\square$

Prime zeta function We next study some more properties of concentrated measures corresponding to series. We particularly consider the prime zeta function ([25],[26]). We recall a definition from [5]. Suppose $f : [0, \pi] \to [0, 1]$ is increasing. Then, a distribution μ on $\mathbb{S}^1$ is said to be f-equitable, if for each $0 \leq \theta' \leq \theta'' \leq 2\pi$ such that $\theta'' - \theta' \leq \pi$, we have that

$$\mu(\{z : \theta' \leq \theta(z) \leq \theta''\}) \leq f(\theta'' - \theta') \tag{2.18}$$

and for $0 \leq \theta' \leq \theta'' \leq 2\pi$ such that $\theta' + (2\pi - \theta'') \leq \pi$,

$$\mu(\{z : \theta(z) \in [0, \theta'] \cup [\theta'', 2\pi]\}) \leq f(\theta' + (2\pi - \theta'')). \tag{2.19}$$

Define $f_0(\theta) = \frac{\theta}{2\pi}$ and $f_1(\theta) = \frac{1}{1 + \cos(\frac{\theta}{2})}$. The uniform measure on the unit circle is the only f_0-equitable measure. Note that μ is f_1-equitable if and only if μ is not concentrated. We will now study f_1-equitable (or simply equitable) distributions for points in the domain of the prime zeta function.

Suppose that $\mathbb{P} \subseteq \mathbb{N}$ is the set of all prime numbers. For each $s = (\sigma, t)$, define the function

$$P(s) = \sum_{p \in \mathbb{P}} \frac{1}{p^s}. \tag{2.20}$$

The function P is the prime zeta function. Now, define μ_s^P to be the associated probability measure on the unit circle $\mathbb{S}^1$ as

$$\mu_s^P(A) := \frac{\sum_{p \in \mathbb{P}: p^{-s} \in A} p^{-\sigma}}{\sum_{p \in \mathbb{P}} p^{-\sigma}}. \tag{2.21}$$

We will first prove the following proposition in planar geometry that will be useful.

Proposition 2.3. *Suppose that $\varphi_1, \varphi_2, \varphi_3 \geq 0$ such that $\varphi_1 + \varphi_2 + \varphi_3 = 1$. Then, $\max\{\varphi_1, \varphi_2, \varphi_3\} \leq 0.5$ if and only if there exist $z_1, z_2, z_3 \in \mathbb{S}^1$ such that*

$$\varphi_1 z_1 + \varphi_2 z_2 + \varphi_3 z_3 = 0. \tag{2.22}$$

Proof. Note that in the plane there exists $z_1, z_2, z_3 \in \mathbb{S}^1$ such that $\varphi_1 z_1 + \varphi_2 z_2 + \varphi_3 z_3 = 0$ if and only if φ_1, φ_2 and φ_3 are the lengths of three sides of a triangle. This, in turn is equivalent to the condition $\max\{\varphi_1, \varphi_2, \varphi_3\} \leq 0.5$. $\qquad \square$

Now, we shall prove the following proposition.

Proposition 2.4. *There exists an interval $[\sigma_*, \sigma^*] \subseteq [1, \frac{7}{4}]$ such that for each $\sigma \in [\sigma_*, \sigma^*]$, there exist countably many pairwise disjoint intervals $\{[\underline{t}_k, \overline{t}_k]\}_{k \in \mathbb{Z}^+}$, such that $\lim_{k \to \infty} \underline{t}_k \to +\infty$ and for each k and $t \in [\underline{t}_k, \overline{t}_k]$, we have that $\mu_{(\sigma,t)}^P$ is f_1-equitable.*

Proof. Let $\sigma^* > 1$ such that

$$\sum_{p \in \mathbb{P}: p \geq 7} \frac{1}{p^{\sigma^*}} = 0.5. \tag{2.23}$$

This follows from the continuity of the prime zeta function. We will now exhibit some necessary calculations. Note that, by an integral comparison,

$$\sum_{p \in \mathbb{P}: p \geq 7} \frac{1}{p^{\frac{7}{4}}} \leq \frac{4}{3(6)^{3/4}} < 0.5 \tag{2.24}$$

Hence, by the monotonicity of the prime zeta function in σ on the real line, $\sigma^* < \frac{7}{4}$. It follows that

$$\frac{1}{2^{\sigma^*}} + \frac{1}{3^{\sigma^*}} + \frac{1}{5^{\sigma^*}} > \frac{1}{2^{\frac{7}{4}}} + \frac{1}{3^{\frac{7}{4}}} + \frac{1}{5^{\frac{7}{4}}} > 0.5. \tag{2.25}$$

From the above derivations in 2.23,2.24 and 2.25 we have that

$$\max\left\{ \frac{2^{-\sigma^*}}{P(\sigma^*)}, \frac{3^{-\sigma^*} + 5^{-\sigma^*}}{P(\sigma^*)}, \frac{\sum_{p \in \mathbb{P}: p \geq 7} p^{-\sigma^*}}{P(\sigma^*)} \right\} < 0.5. \tag{2.26}$$

In the above maximum, the three numbers add up to one. Hence, from Proposition 2.3, there exist points $z_A, z_B, z_C \in \mathbb{S}^1$ such that

$$\frac{2^{-\sigma^*}}{P(\sigma^*)} z_A + \frac{3^{-\sigma^*} + 5^{-\sigma^*}}{P(\sigma^*)} z_B + \frac{\sum_{p \in \mathbb{P}: p \geq 7} p^{-\sigma^*}}{P(\sigma^*)} z_C = 0. \tag{2.27}$$

Suppose, for large m, we now consider three disjoint sets $A = \{2^{\sigma^*}(2)^{-(\sigma^*,t)}\}$, $B = \{3^{\sigma^*}(3)^{-(\sigma^*,t)}, 5^{\sigma^*}(5)^{-(\sigma^*,t)}\}$ and $C = \{p^{\sigma^*}(p)^{-(\sigma^*,t)}\}_{p \in \mathbb{P}: 7 \leq p \leq m}$. Note that $A, B, C \subseteq \mathbb{S}^1$. Since for $m \geq 3$, the collection of numbers $\{\frac{\ln(p)}{2\pi}\}_{p \in \mathbb{P}: p \leq m}$ is linearly independent over $\mathbb{Q}$, we apply the multidimensional Weyl criterion ([24]) to the sequence $\left\{\{t\frac{\ln(p)}{2\pi}\}_{p \in \mathbb{P}: p \leq m}\right\}_{t \in \mathbb{N}}$ to position A arbitrarily close to z_A, B arbitrarily close to z_B, and C arbitrarily close to z_C, by varying t. Since m is large, the norm of the tail would be negligible, so that we would approximate the zero mean distribution on the three points z_A, z_B, z_C on the unit circle with probability weights as in 2.26. Lastly, we choose $\sigma_* < \sigma^*$ such that σ_* is close to σ^*, so that the appropriate strict inequalities from above would be satisfied. $\qquad \square$

We also have the following proposition.

Proposition 2.5. *There exists an interval $[\sigma_*, \sigma^*] \subseteq [1, \frac{7}{4}]$ such that for each $\sigma \in [\sigma_*, \sigma^*]$, there exists an infinite subset of primes $T \subseteq \mathbb{P}$ such that for each $t \in T$, we have that $\mu_{(\sigma,t)}^P$ is f_1-equitable.*

Proof. This again follows from the linear independence of $\{\frac{\ln(p)}{2\pi}\}_{p \in \mathbb{P}: p \leq m}$ over $\mathbb{Q}$. Further, we also need the fact from [27] that the sequence $\{\alpha p\}_{p \in \mathbb{P}}$ is uniformly distributed mod 1, for irrational α. See also [24] and [28]. $\qquad \square$

One may remark that in the above propositions, we present uncountably many f_1-equitable points. By appealing to the concentration phenomena in associated measures over the unit circle, we are able to concentrate the measure on any possible arc with span less than π, by varying t for the prime zeta function. This would also allow us to apply the argument principle on triangular contours, where the ends points would be concentrated on polar opposite arcs, ensuring a positive argument/angular change. Such an approach would generate countably many zeros and hence, countably many f_1-equitable points. In contrast, the above proposition constructs approximate zeros as we approximate a zero mean distribution on three points on the circle.

The concentration on any arc (not necessarily around $(1, 0)$) is possible because of the linear independence of $\{\frac{\ln(p)}{2\pi}\}_{p \in \mathbb{P}: p \leq m}$ over $\mathbb{Q}$. Hence, equidistribution follows allowing inhomogenous Diophantine approximation with intercepts. This is in contrast with the Riemann zeta function $\zeta(s) = \sum_{n \geq 1} n^{-s}$ on $\sigma > 1$, for which the associated collection $\{\frac{\ln(n)}{2\pi}\}_{n \in \mathbb{N}: 2 \leq n \leq m}$ is linearly dependent over $\mathbb{Q}$. However, it is possible to consider intercepts which are proportional to the real numbers to be simultaneously approximated.

3 Conclusion

We have shown some new results, which extends prior research. By placing hypotheses on the various mathematical quantities of interest, we study the occurence of concentrated and equitable points, which is a dichotomy related to the non-zero/zero dichotomy of the relevant domain. We have obtained propositions concerning distributional properties that occur infinitely often along the imaginary axis. This presents new insights into the problem studied in the paper.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Riemann B. On the number of prime numbers under a given size. Complete mathematical works and scientific papers. 1859;2:145-155.

[2] Stein EM, Shakarchi R. Complex Analysis. Vol 2. Princeton University Press; 2010.

[3] Basu P. On the Riemann Hypothesis. Asian Journal of Probability and Statistics. 2022;20(4):51-56.

[4] Basu P. On propositions pertaining to the riemann hypothesis. Asian Journal of Probability and Statistics. 2023;21(2):16-21.

[5] Basu P. On Propositions pertaining to the Riemann hypothesis II. Asian Journal of Probability and Statistics. 2023;25(2):63-74.

[6] Mangoldt H von. On the number of prime numbers under a given size. Complete mathematical works and scientific papers. 1905;61:1-19.

[7] Hardy GH. On the zeros of the function (s) the Riemann. CR Acad Sci Paris. 1914;158(1914):1012.

[8] The zeros of Riemann's zeta-function on the critical line. Mathematische Zeitschrift. 1921;10(3):283-317.

[9] Conrey JB. The riemann hypothesis. Notices of the AMS. 2003;50(3):341-353.

[10] Lagarias JC. An elementary problem equivalent to the Riemann hypothesis. The American Mathematical Monthly. 2002;109(6):534-543.

[11] Bump D, Choi KK, Kurlberg P, Vaaler J. A local Riemann hypothesis, I. Mathematische Zeitschrift. 2000;233(1):1-18.

[12] Borwein P, Choi S, Rooney B, Weirathmueller A. The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike. Springer; 2008.

[13] Platt D, Trudgian T. The Riemann hypothesis is true up to 3·10 12. Bulletin of the London Mathematical Society. 2021;53(3):792-797.

[14] Nicolas JL. The sum of divisors function and the Riemann hypothesis. The Ramanujan Journal. Published online 2021:1-45.

[15] Johnston DR. Improving bounds on prime counting functions by partial verification of the Riemann hypothesis. The Ramanujan Journal. 2022;59(4):1307-1321.

[16] Maynard J, Pratt K. Half-isolated zeros and zero-density estimates. arXiv preprint arXiv:2206 11729. Published online 2022.

[17] Guth L, Maynard J. New large value estimates for Dirichlet polynomials. arXiv preprint arXiv:2405 20552. Published online 2024.

[18] Pierpont J. Functions of a Complex Variable. Ginn; 1914.

[19] Lindelof E. Quelques Remarques Sur La Croissance de La Fonction [Zeta](s). Gauthier-Villars; 1908.

[20] Schrijver A. Theory of Linear and Integer Programming. John Wiley Sons; 1998.

[21] Boyd S, Vandenberghe L. Convex Optimization. Cambridge university press; 2004.

[22] Conforti M, Cornuéjols G, Zambelli G. Integer Programming Models. Springer; 2014.

[23] Schmidt WM. Diophantine Approximation. Springer Science Business Media; 1996.

[24] Kuipers L, Niederreiter H. Uniform Distribution of Sequences. Courier Corporation; 2012.

[25] Glaisher JWL. On the sums of inverse powers of the prime numbers. Quart J Math. 1891;25:347-362.

[26] Fröberg CE. On the prime zeta function. BIT Numerical Mathematics. 1968;8:187-202.

[27] Vinogradov IM. He en estimate of trigonometric sums with prime number. Proceedings of the Russian Academy of Sciences Mathematical Series. 1948;12(3):225-248.

[28] Harman G. Diophantine approximation by prime numbers. Journal of the London Mathematical Society. 1991;2(2):218-226.

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