**Asian Research Journal of Mathematics**

**Volume 20, Issue 5, Page 50-54, 2024; Article no.ARJOM.114784**
*ISSN: 2456-477X*

# Equal Sums of Four Even Powers

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***Author's contribution***

*The sole author designed, analysed, interpreted and prepared the manuscript.*

***Article Information***

DOI: https://doi.org/10.9734/arjom/2024/v20i5802

**Open Peer Review History:**
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/114784

***Received: 11/02/2024***
***Accepted: 15/04/2024***
***Published: 06/06/2024***

***Original Research Article***

## Abstract

Let $u, v, w, z, k, m$ and $I$ be any integers such that $k = z - w = w - v = v - u$. The study of integer $I$ for which $I = u^{2n} + v^{2n} + w^{2n} + z^{2n} = k^2 + m^2 + r^2$ is not known. This study is therefore, set to partially overcome this challenge by introducing new formula relating sums of four even powers as an exact sum of three squares.

*Keywords: Keyword, sums of three squares, sums of four even powers.*

## 1 Introduction

The study of integer decomposition into sums of powers is an area that has received much attention since the advent of cryptography . Perhaps, this is because of the fact that integer factorization has direct application in cryptography. Most researchers who attempted the representations of integers as as a sum of powers has met

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**Cite as:** *Simatwo, Kimtai Boaz. 2024. "Equal Sums of Four Even Powers". Asian Research Journal of Mathematics 20 (5):50-54. https://doi.org/10.9734/arjom/2024/v20i5802.*

their effort with very minimal success. For recent work on polynomial equations on sums of powers see [1-11] and for detailed recap on integer sums of two square studies the reader may refer to [12-16]. In most of this studies, the literature on integer representation as a sum of four powers is still hardily available. Moreover, documented results on the relationship between sums of four powers and and sums of three squares is not known. This study is therefore, set to introduce and develop the formula $I = u^{2n} + v^{2n} + w^{2n} + z^{2n} = k^2 + m^2 + n^2$ which has also has geometrical application in field of Geometry.

# 2 Main Results

**Theorem 2.1.** *$I = u^{2u} + v^{2u} + w^{2u} + z^{2u} = k^2 + m^2 + n^2$ has solution in integers if $v^u = u^u + k, w^u = u^u + 2k, z^u = u^u + 3k$ and $h \geq 1$.*

*Proof.* Suppose $v^u = u^u + k, w^u = u^u + 2k, z^u = u^u + 3k$. Then, $u^{2u} + v^{2u} + w^{2u} + z^{2u} = u^{2u} + (u^u + k)^2 + (u^u + 2k)^2 + (u^u + 3k)^2 = u^{2u} + u^{2u} + 2u^u k + k^2 + u^{2u} + 4u^u k + 4k^2 + u^{2u} + 6u^u k + 9k^2 = 4u^{2u} + 12u^u k + 14k^2 = k^2 + 4u^2 + 12u^u k + 9k^2 + 4k^2 = k^2 + (2u^u + 3k)^2 + (2k)^2 \cdots (**)$. Letting $m = 2u + 3k$ and $n = 2k$ we obtain $I = u^{2u} + v^{2u} + w^{2u} + z^{2u} = k^2 + m^2 + n^2$ completing the proof. This clearly shows that sums of four squares can be suppressed as a sum of three squares. □

Application of theorem 2.1.

Suppose a farmer has four square plots and will wish to share them among his three children so that each child can get a squares plot . How does the farmer go about it?. The answer to this question easily follows from theorem 2.1. This theorem can also be extended to other real life application.

**Theorem 2.2.** *$I = u^{2u} + v^{2u} + w^{2u} + z^{2u} = k^2 + m^2 + n^2$ has solution in integers if $v^{2u} = u^{2u} + k, w^{2u} = u^{2u} + 2k, z^{2u} = u^{2u} + 3k$ and $h \geq 1$ .*

*Proof.* Suppose $v^{2u} = u^{2u} + k, w^{2u} = u^{2u} + 2k, z^{2u} = u^{2u} + 3k$. Then, $u^{2u} + v^{2u} + w^{2u} + z^{2u} = u^{4u} + (u^{2u} + k)^2 + (u^{2u} + 2k)^2 + (u^{2u} + 3k)^2 = 4u^{4u} + 14k^2 + 12ku^{2u} = k^2 + (2u^{2u} + 3k)^2 + 4k^2$. Put $m = 2u^{2u} + 3k$ and $n = 2k$ we have $I = u^{2u} + v^{2u} + w^{2u} + z^{2u} = k^2 + m^2 + n^2$ concluding the proof. □

The result on theorem 2.2 shows that sums of four fourth powers can be suppressed as a sum of three squares.

**Theorem 2.3.** *$I = u^{3u} + v^{3u} + w^{3u} + z^{3u} = k^2 + m^2 + n^2$ has solution in integers if $v^{2u} = u^{2u} + k, w^{2u} = u^{2u} + 2k, z^{2u} = u^{2u} + 3k$ and $h \geq 1$.*

*Proof.* Suppose $v^{2u} = u^{3u} + k, w^{2u} = u^{3u} + 2k, z^{2u} = u^{3u} + 3k$. Then, $u^{3u} + v^{3u} + w^{3u} + z^{3u} = u^{3u} + (u^{3u} + k)^2 + (u^{3u} + 2k)^2 + (u^{3u} + 3k)^2 = 4u^{6u} + 12ku^{3u} + 14k^2 = k^2 + (2u^{3u} + 3k)^2 + 4k^2$. Put $m = (2u^{3u} + 3k)$ and $n = 2k$ so that $I = u^{3u} + v^{3u} + w^{3u} + z^{3u} = k^2 + m^2 + n^2$ concluding the proof. □

The result on theorem 2.3 shows that sums of four sixth powers powers can be suppressed as a sum of three squares.

**Theorem 2.4.** *$I = u^{4u} + v^{4u} + w^{4u} + z^{4u} = k^2 + m^2 + n^2$ has solution in integers if $v^{2u} = (u^{4u} + k)^2, w^{2u} = (u^{4u} + 2k)^2, z^{2u} = (u^{4u} + 3k)^2$ .*

*Proof.* Suppose $v^{2u} = u^{4u} + k, w^{2u} = u^{4u} + 2k, z^{2u} = u^{4u} + 3k$. Then, $u^{4u} + v^{4u} + w^{4u} + z^{4u} = u^{8u} + (u^{4u} + k)^2 + (u^{4u} + 2k)^2 + (u^{4u} + 3k)^2 = 4u^{8u} + 12ku^{4u} + 14k^2 = k^2 + (2u^{4u} + 3k)^2 + 4k^2$. Put $m = (2u^{4u} + 3k)$ and $n = 2k$ so that $I = u^{4u} + v^{4u} + w^{4u} + z^{4u} = k^2 + m^2 + n^2$ establishing the proof. □

The result on theorem 2.4 shows that sums of four eighth powers can be suppressed as a sum of three squares.

**Theorem 2.5.** *$I = u^{2n} + v^{2n} + w^{2n} + z^{2m} = k^2 + m^2 + n^2$ has solution in integers if $v^2 = u^n + k, w^2 = u^n + 2k, z^2 = u^n + 3k$ .*

*Proof.* Suppose $v^2 = u^n + k, w^2 = u^n + 2k, z^2 = u^n + 3k$ . Then, $u^{2n} + v^{2n} + w^{2n} + z^{2n} = u^{2n} + (u^n + k)^2 + (u^n + 2k)^2 + (u^n + 3k)^2 = 4u^{2n} + 14k^2 + 12ku^n = k^2 + (2u^n + 3k)^2 + 4k^2$. Put $m = (2u^n + 3k)$ and $n = 2k$ so that $I = u^{2n} + v^{2n} + w^{2n} + z^{2m} = k^2 + m^2 + n^2$ establishing the proof. □

The result on theorem 2.3 shows that sums of four even powers powers can be suppressed as a sum of three squares.

## 2.1 Some examples

In this subsection, we provide some examples to argument our results in Theorem 2.1.

**Table 1. Results of Theorem 2.1**

| $u^2$ | $v^2$ | $w^2$ | $z^2$ | $u^2 + v^2 + w^2 + z^2 = I = k^2 + m^2 + n^2$ | $k^2$ | $m^2$ | $n^2$ |
|---|---|---|---|---|---|---|---|
| 1 | 4 | 9 | 16 | 30 | 1 | 25 | 4 |
| 4 | 9 | 16 | 25 | 54 | 1 | 49 | 4 |
| 1 | 9 | 25 | 49 | 84 | 4 | 64 | 16 |
| 4 | 25 | 64 | 121 | 214 | 9 | 169 | 36 |
| 9 | 49 | 121 | 225 | 404 | 16 | 324 | 64 |
| 25 | 100 | 225 | 400 | 750 | 25 | 625 | 100 |
| 4 | 64 | 196 | 400 | 664 | 36 | 484 | 144 |
| 16 | 121 | 324 | 625 | 1086 | 49 | 841 | 196 |
| 49 | 225 | 529 | 961 | 1764 | 64 | 1444 | 256 |

In this subsection, we provide some examples to argument our results in Theorem 2.2.

**Table 2. Results in Theorem 2.2**

| $u^4$ | $v^4$ | $w^4$ | $z^4$ | $u^4 + v^4 + w^4 + z^4 = I = k^2 + m^2 + n^2$ | $k^2$ | $m^2$ | $n^2$ |
|---|---|---|---|---|---|---|---|
| 1 | 4 | 9 | 16 | 30 | 1 | 25 | 4 |
| 16 | 25 | 36 | 49 | 126 | 1 | 121 | 4 |
| 81 | 121 | 169 | 225 | 596 | 4 | 576 | 16 |
| 256 | 400 | 576 | 784 | 2018 | 16 | 1986 | 16 |
| 625 | 784 | 961 | 1156 | 3526 | 9 | 3481 | 36 |
| 16 | 49 | 100 | 169 | 334 | 9 | 225 | 36 |
| 1 | 36 | 121 | 256 | 441 | 25 | 289 | 100 |
| 81 | 144 | 225 | 324 | 774 | 9 | 729 | 36 |
| 256 | 324 | 400 | 484 | 1464 | 4 | 1444 | 16 |

In this subsection, we provide some examples to argument our results in Theorem 2.3.

In this subsection, we provide some examples to argument our results in Theorem 2.4.

**Table 3. Results in Theorem 2.3**

| $u^6$ | $v^6$ | $w^6$ | $z^6$ | $u^6+v^6+w^6+z^6=I=k^2+m^2+n^2$ | $k^2$ | $m^2$ | $n^2$ |
|---|---|---|---|---|---|---|---|
| $5^6$ | $8^6$ | $7^6$ | $8^6$ | 442074 | 1 | 442069 | 4 |
| $6^6$ | $7^6$ | $8^6$ | $9^6$ | 957890 | 1 | 957885 | 4 |
| $9^6$ | $11^6$ | $13^6$ | $15^6$ | 18520436 | 4 | 18520416 | 16 |
| $14^6$ | $17^6$ | $20^6$ | $23^6$ | 243702994 | 9 | 243702949 | 36 |
| $19^6$ | $23^6$ | $27^6$ | $31^6$ | 1470005940 | 16 | 1470005860 | 64 |
| $25^6$ | $30^6$ | $35^6$ | $40^6$ | 6907406250 | 36 | 6907406070 | 144 |
| $26^6$ | $32^6$ | $38^6$ | $44^6$ | $1.1649...*10^{10}$ | 36 | $1.1649...*10^{10}$ | 144 |
| $32^6$ | $39^6$ | $46^6$ | $53^6$ | $3.6231...*10^{10}$ | 49 | $3.6231...*10^{10}$ | 196 |
| $39^6$ | $47^6$ | $55^6$ | $63^6$ | $1.0450...*10^{11}$ | 64 | $1.0450...*10^{11}$ | 256 |

**Table 4. Results in Theorem 2.4**

| $u^8$ | $v^8$ | $w^8$ | $z^8$ | $u^8+v^8+w^8+z^8=I=k^2+m^2+n^2$ | $k^2$ | $m^2$ | $n^2$ |
|---|---|---|---|---|---|---|---|
| $5^8$ | $6^8$ | $7^8$ | $8^8$ | 24612258 | 1 | 2461253 | 4 |
| $8^8$ | $9^8$ | $10^8$ | $11^8$ | 374182813 | 1 | 374182813 | 4 |
| $17^8$ | $19^8$ | $21^8$ | $23^8$ | $1.4009...*10^{11}$ | 4 | $1.4009...*10^{11}$ | 16 |
| $32^8$ | $36^8$ | $40^8$ | $44^8$ | $2.4522...*10^{13}$ | 16 | $2.4522...*10^{13}$ | 16 |
| $37^8$ | $40^8$ | $43^8$ | $46^8$ | $3.8293...*10^{13}$ | 9 | $3.8293...*10^{13}$ | 36 |
| $16^8$ | $19^8$ | $22^8$ | $25^8$ | $2.2874...*10^{11}$ | 9 | $2.2874...*10^{11}$ | 36 |
| $21^8$ | $26^8$ | $31^8$ | $36^8$ | $3.9206...*10^{12}$ | 25 | $3.9206...*10^{12}$ | 100 |
| $21^8$ | $24^8$ | $27^8$ | $30^8$ | $1.0864...*10^{12}$ | 9 | $1.0864...*10^{12}$ | 36 |
| $24^8$ | $26^8$ | $28^8$ | $30^8$ | $1.3525...*10^{12}$ | 4 | $1.3525...*10^{12}$ | 16 |

**Conjecture 1.** *$I = u^{2n} + v^{2n} + w^{2n} + z^{2m} = k^2 + m^2 + n^2$ is impossible if $k \neq z - w \neq w - v \neq v - u$ for some integer $k$.*

# 3 Conclusion

This research, has contributed to problem of integer representation of sums of four powers as a sum of three squares. However,the problem of integer representation as a sum of powers is still a very long standing problem. We therefore, encourage other researchers to devote there attention in this field of research.

# Acknowledgements

The authors would like to thank the anonymous reviewers for carefully reading the article and for their helpful comments.

# Competing Interests

Author has declared that no competing interests exist.

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