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A Stabilization for a Coupled Wave System with Nonlinear and Arbitrary Damping

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

Aims/ Objectives: In this paper, based on motivations coming from various physical applications, we consider a coupled system of the wave in a one-dimensional bounded domain with nonlinear localized damping acting in their equations. We also discuss the well-posedness and smoothness of solutions using the nonlinear semigroup theory. Then, we give the asymptotic stability and rates decay to the coupled system, based on solution of an ordinary differential equation, since the feedback functions and the localized functions satisfy some properties widely treated in obtaining uniform decay rates for solutions of semilinear wave equation. Furthermore, the result requires the obtaining of the internal observability inequality for the conservative system.

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1 Introduction

Let us consider the system

$$\begin{cases} u_{tt} - u_{xx} + \xi_1(x)g_1(u_t) + \alpha v = 0 \\ v_{tt} - v_{xx} + \xi_2(x)g_2(v_t) + \alpha u = 0, \end{cases} \tag{1.1}$$

in $(0, L) \times (0, T)$, with Dirichlet boundary conditions

$$u(0, t) = 0, u(L, t) = 0, v(0, t) = 0, v(L, t) = 0, t \in (0, T) \tag{1.2}$$

and initial conditions

$$u(x, 0) = u_0, u_t(x, 0) = u_1, v(x, 0) = v_0, v_t(x, 0) = v_1, x \in (0, L) \tag{1.3}$$

The positive constant α is a coupling parameter, related to the approximation of membranes as described in the model above. This model represents the evolution of a system that involves two elastic membranes subject to an elastic force, firstly presented in [1].

Later, the nonnegative localizing functions ξ_i will be supposed to belong to $L^\infty(0, L)$ while the functions g_i will be supposed to be continuous and monotone.

For a single wave equation, it is important to mention two works: In [2] a result based on microlocal techniques and geometric optics analysis allows us to find geometrical characterization of control location and minimal control time in the exact controllability of waves. After eventual reflection, diffraction, or sliding on the boundary, every optic ray emitted from the observation domain has to reach the control zone. This is a necessary and sufficient condition to obtain exponential decay, namely, the damping region satisfies the well-known Geometric Control Condition(GCC). For the wave equation an observability inequality requires of the GCC, sufficiently large time and geometric conditions on the subset to absorb all rays of geometric optics. Another important result is found in [3], when the feedback term depends on the velocity in a linear way, which proves that the energy related to equation decays exponentially if the damping region contains a neighborhood of the boundary Γ or, at least, contains a neighborhood ω of the particular part given by $\{x \in \Gamma : (x - x_0).\nu(x) \leq 0\}$. For systems, there are in literature some works with $\Omega \subset \mathbb{R}^n$, as [4] or [5], in the context of linear damping mechanisms. In [4], the abstract model is

$$\begin{cases} u_1'' + A_1 u_1 + B u_1' + \alpha P u_2 = 0 \text{ in } V_1', \\ u_2'' + A_2 u_2 + \alpha P^* u_1 = 0 \text{ in } V_2', \\ (u_1', u_1')(0) = (u_1^0, u_1^1) = U_1^0 \in V_1 \times H, \\ (u_2', u_2')(0) = (u_2^0, u_2^1) = U_2^0 \in V_2 \times H, \end{cases}$$

where H, $V_1 \subset H$, and $V_2 \subset H$ are separable Hilbert spaces; A_1, A_2 are coercive self-adjoint unbounded operators in H; B is unbounded symmetric in H, whereas the coupling operator P is assumed to be bounded in H; P^* is the adjoint operator of P. Under a condition on the operators of each equation and on the boundary feedback operator, the energy of smooth solutions of that system decays polynomially at ∞, and them apply this abstract result to several systems as wave-wave systems, Kirchhoff-Petrowsky systems, and wave-Petrowsky systems. On the other hand in [5], we have the system

$$\begin{cases} u'' + A_1 u + B u' + \alpha v = 0 \\ v'' + A_2 v + \alpha u = 0 \end{cases}$$

in a separable Hilbert space H, where A_1, A_2 and B are self-adjoint positive linear operators in H, and in addition, B to be a bounded operator. The solutions decay polynomially at infinity, and that this decay rate is, in some sense, optimal. The stabilization result for abstract evolution equations, studied by them is also applied to study the asymptotic behaviour of various systems of partial differential equations. Furthermore, many questions about abstract systems are given by them. If waves propagate at different speeds(i.e. variable coefficients) the situation becomes more delicate and the strong stability is not true in general. Furthermore, we limit our attention to one-dimensional domain. On the other hand, we have not been able to prove this result for variable coefficients, as well as knowing how the regularity of the coefficients affects the stabilization properties, in the same spirit of recent papers [6, 7, 8, 9]. To our knowledge, the problem remains open for systems.

The prime object of study in this paper is to show the standpoint of another rate of decay for the (wave-wave) system with frictional damping mechanisms. A distinctive feature of the above mentioned paper is exactly to consider that these mechanisms will be nonlinear localized mechanisms. Although, we are stabilizing the two equations with internal damping, by virtue of the used method. As far as we are concerned, this is the first work which establishes this result for nonlinear damping mechanisms. This is accomplished by following multipliers technique and the method presented in [10], where without any geometrical condition and without assuming that the feedback has a polynomial growth in zero, they showed that the energy decays as fast as the solution of some associated differential equation. This rate will be then:

$$E(t) \leq S\left(\frac{t}{T_0} - 1\right) E(0) \searrow 0, \text{ for all } t \geq T_0,\ t \to \infty,$$

for energy $E(t)$ of the system (1.1), where the scalar function $S(t)$ (nonlinear contraction) is the solution of an ODE. Still we use an adaptation of [11], where they generate appropriate estimates for the energy functional as opposed to the classical method of constructing a particular Lyapunov function for a general nonlinear equation.

In other words, we obtain exponential decay rates for the damping that is bounded from below by a linear function and algebraic decay rates for polynomially decaying dissipation at the origin.

This paper is structured as follows. First, in the Section 2, we present the assumptions and the result of the existence of solutions to the proposed system. In the Section 3 we obtain an essential inequality for inhomogeneous system. Finally, in the Section 4, we established the rates decay for the solutions of the nonlinear localized damped system.

2 Assumptions and Existence

Let's assume that α is a real number such that $|\alpha|$ be a sufficiently small positive quantity, so that we have a positive energy. Since $\{u, v\}$ is a solution of (1.1) the energy of the system related to this solution is

$$E(t) = \frac{1}{2}\int_0^L \left(|u_t|^2 + |u_x|^2 + |v_t|^2 + |u_x|^2 + 2\alpha uv\right)dx, \tag{2.1}$$

with t nonnegative.

Assumptions around the mechanisms of damping of the problem will be made. Firstly, to accomplish the decay rates we will assume that

H1: The feedback function g_i, for each $i = 1, 2$, is continuous and monotone increasing, and, in addition, satisfies the following conditions:

(i) $g_i(s)s > 0$ for $s \neq 0$,

(ii) $k_i\, s \leq g_i(s) \leq K_i\, s$ for $|s| > 1$,

where k_i and K_i are positive constants, and $k_i \leq K_i$.

And now, about the geometrical condition we will assume the following effective damping region:

H2: Let ε be a sufficiently small positive quantity. Assume that $\xi_i \in L^\infty(0, L)$ are nonnegative functions such that

$$\xi_i(x) \geq \tau_i > 0 \text{ a.e. in } I_\varepsilon := (L - \varepsilon, L),\ i = 1, 2. \tag{2.2}$$

where τ_i are positive constants.

Lemma 2.1. *The energy functional* (2.1) *satisfies*

$$\frac{dE(t)}{dt} = -\int_0^L \{\xi_1(x)g_1(u_t)u_t + \xi_2(x)g_2(v_t)v_t\}\, dx \leq 0, \forall t \geq 0. \tag{2.3}$$

Now consider the Hilbert space

$$\mathcal{H} := H_0^1(0, L) \times H_0^1(0, L) \times L^2(0, L) \times L^2(0, L), \tag{2.4}$$

with internal product

$$\langle U, V\rangle_{\mathcal{H}} = \int_0^L \{\nabla u_1 \nabla v_1 + \nabla u_2 \nabla v_2 + \alpha(u_1 v_2 + u_2 v_1) + u_3 v_3 + u_4 v_4\}\, dx,$$

with $U = (u_1, u_2, u_3, u_4)^{\mathbb{T}}$ and $U = (v_1, v_2, v_3, v_4)^{\mathbb{T}}$, where to make use of $\mathbb{T}$ to denote the transpose.

If we denote $W(t) = \{u, v, u_t, v_t\}^{\mathbb{T}}$ then the initial boundary value problem (1.1)-(1.3) can be rewritten as a first order problem as follows:

$$\begin{cases} \dfrac{dW}{dt}(t) + \mathcal{A}W(t) = 0 \\ W(0) = W_0, \end{cases} \tag{2.5}$$

where $W_0 = \{u_0, v_0, u_1, v_1\}$ and the operator $\mathcal{A} = D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by $\mathcal{A} = -(A + B)$ with component operators defined by

$$D(A) = (H_0^1(0, L) \cap H^2(0, L))^2 \times (H_0^1(0, L))^2 \quad \text{and} \quad D(B) = \mathcal{H},$$

and

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Delta & -\alpha I & 0 & 0 \\ -\alpha I & \Delta & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\xi_1(x)g_1(.) & 0 \\ 0 & 0 & 0 & -\xi_2(x)g_2(.) \end{pmatrix}.$$

We observe that, in this case we have $D(\mathcal{A}) = D(A)$. From nonlinear semigroups theory [12], where X will be a real Banach space and X^* will denote its dual space, we have the following:

Definition 2.1. A set $T \subset X \times X^*$ is called monotone if

$$(x_1 - x_2, y_1 - y_2) \geq 0,$$

for any element $[x_i, y_i]$ of T, $i = 1, 2$. If T is a single-valued operator from $X \times X^*$, then the monotonicity condition becomes

$$(x_1 - x_2, Tx_1 - Tx_2) \geq 0,$$

for all $x_1, x_2 \in D(T)$.

Definition 2.2. A monotone subset of $X \times X^*$ is said to be maximal monotone if it is not properly contained in any other monotone subset of $X \times X^*$.

Definition 2.3. The operator $T : X \to X^*$ is said to be bounded if it maps every bounded subset of X into a bounded subset of X^*.

Definition 2.4. Let T be a single-valued operator defined from X to X^* such that D(T) = X. T is said to be hemicontinuous on X if

$$\omega - \lim_{t \to 0} T(x + ty) = Tx,$$

for any x and y in X, where $\omega - \lim$ denotes the limit in the weak topology.

And from [13] we have the following existence and regularity result:

Theorem 2.2. *Let Assumptions* **H1** *and* **H2** *hold. For any $W_0 \in D(\mathcal{A})$ there will exist a unique strong solution for* (2.5). *Furthermore, if $W_0 \in \mathcal{H}$ then* (2.5) *will admit a unique weak solution.*

Proof. We need to show that the operator $\mathcal{A} = -(A + B)$ is maximal monotone, after, we will use Brézis[13, Theorem 3.1] in order to conclude the result. At this moment we shall divide our proof into two parts, because we will use Barbu[12, Corollary 1.1], where it is enough to show that:

(i) The operator $-A$ is maximal monotone.

(ii) $-B$ is monotone, hemicontinuous, and a bounded operator.

Proof of (i). Here we show that $-A$ is monotone and $\mathcal{R}(I - A) = \mathcal{H}$, so by [13, Proposition 2.2] the result follow. In fact, as

$$(-AW, W)_{\mathcal{H}} = 0$$

we have that $-A$ is monotone. Now considering $(F_1, F_2, F_3, F_4) \in \mathcal{H}$, we obtain

$$u - u_t = F_1 \in H_0^1(0, L), \tag{2.6}$$

$$v - v_t = F_2 \in H_0^1(0, L), \tag{2.7}$$

$$u_t - (u_{xx} - \alpha v) = F_3 \in L^2(0, L), \tag{2.8}$$

$$v_t - (v_{xx} - \alpha u) = F_4 \in L^2(0, L), \tag{2.9}$$

and then

$$u - (u_{xx} - \alpha v) = f_1 = F_1 + F_3 \in L^2(0, L),$$
$$v - (v_{xx} - \alpha u) = f_2 = F_2 + F_4 \in L^2(0, L).$$

This way we are able to see that

$$a : (H_0^1(0, L) \times H_0^1(0, L))^2 \longrightarrow \mathbb{R}$$

defined by

$$\int_0^L \{u_x \nabla\theta_1 + v_x \nabla\theta_2 + u\theta_1 + v\theta_2\}\, dx$$

where we have that

$$(u,v),(\theta_1,\theta_2) \in (H_0^1(0,L) \times H_0^1(0,L))$$

is a bilinear form, continuous and coercive. From the Lax-Milgram theorem, we obtain the desired surjection.

Therefore, the operator $-A$ is maximal monotone by [13, Proposition 2.2].

(ii) $-B$ is monotone, hemicontinuous, and a bounded operator.

Now, using Assumptions **H1** and **H2**, the operator $-B$ satisfies

$$(-BW, W)_{\mathcal{H}} \geq 0.$$

Then it is monotone. Taking

$$W_i = (u_i, v_i, \bar{u}_i, \bar{v}_i)^{\mathbb{T}} \in \mathcal{H},\ i = 1, 2,$$

where prime is used for the partial derivatives with respect to time $t \geq 0$, we consider the following expression

$$(-B(W_1 + tW_2), W)_{\mathcal{H}} \quad = \quad \left(\xi_1(x) g_1(\bar{u}_1 + t\bar{u}_2), u'\right)_{L^2} + \left(\xi_2(x) g_2(\bar{v}_1 + t\bar{v}_2), v'\right)_{L^2},\ t > 0.$$

We desire to show that

$$\begin{cases} \lim\limits_{t\to 0} \left(\xi_1(x) g_1(\bar{u}_1 + t\bar{u}_2), u'\right)_{L^2} = \left(\xi_1(x) g_1(\bar{u}_1), u'\right)_{L^2}, \\ \lim\limits_{t\to 0} \left(\xi_2(x) g_2(\bar{v}_1 + t\bar{v}_2), v'\right)_{L^2} = \left(\xi_2(x) g_2(\bar{v}_1), v'\right)_{L^2}. \end{cases} \tag{2.10}$$

We will show only the first equality, because the other is analogous. For this, let $f \in L^1(0,L)$ be defined by

$$f(x) = \left(\xi_1(x) g_1(\bar{u}_1(x)), u'(x)\right),$$

and defining the sequence $(f_n) \subset L^1(0,L)$, given by

$$f_n(x) = \xi_1(x) g_1(\bar{u}_1(x) + \frac{1}{n}\bar{u}_2(x)) u(x),$$

we have,

$$\lim_{n\to\infty} f_n(x) = f(x),$$

almost ever in $(0,L)$. Taking into account the set

$$X_n = \left\{ x \in [0,L];\ |\bar{u}_1(x) + \frac{1}{n}\bar{u}_2(x)| < 1 \right\},$$

we obtain the sequence (f_n) limited. Then, by Lebesgue's Dominated Convergence Theorem we obtain the desired limit of the first equality from (2.10). Therefore, $-B$ is hemicontinuous.

Finally, the operator $-B$ maps every bounded subset into a bounded subset. In fact, if $||W||_{\mathcal{H}} < M$ for any $M > 0$, using the assumption **H1** we have

$$|| - BW||_{\mathcal{H}} \quad \leq \quad ||\beta_1||_\infty^2 \{L. \sup_{s\in[-1,1]} |g_1(s)|^2 + K_1.M\} + ||\beta_2||_\infty^2 \{L. \sup_{s\in[-1,1]} |g_2(s)|^2 + K_2.M\}.$$

With the statements (i) and (ii), the operator $\mathcal{A}$ is maximal monotone of $\mathcal{H}$, and so from [13, Theorem 3.1] we conclude the proof. □

Before presenting our main result, we need a useful resource used in his proof given at next section. In fact, for the sake of clarifying the resolution firstly we recommend getting the observability for the conservative system.

3 Observability

Recall that for the undamped system

$$\begin{cases} u_{tt} - u_{xx} + \alpha v = 0, \\ v_{tt} - v_{xx} + \alpha u = 0 \end{cases} \tag{3.1}$$

in $(0, L) \times (0, T)$, with Dirichlet boundary conditions (1.2) and initial conditions (1.3), we have a conservative system. In order to prove the observability for the system (1.1):

$$\begin{cases} u_{tt} - u_{xx} + \xi_1(x)g_1(u_t) + \alpha v = 0 \\ v_{tt} - v_{xx} + \xi_2(x)g_2(v_t) + \alpha u = 0, \end{cases}$$

we shall work with regular solutions; however they remain valid for weak solutions by using density arguments. Multiplying the first equation of the system (1.1) by xu_x, the second equation of the system (1.1) by xv_x and performing an integration by parts we arrive at

$$\begin{aligned} 0 &= \frac{1}{2}\int_0^T \int_0^L u_t^2 + v_t^2 + |u_x|^2 + |v_x|^2 \, dxdt \\ &- \frac{L}{2}\int_0^T |u_x|^2(L) + |v_x|^2(L)dt + \left[\int_0^L u_t x u_x + v_t x v_x \, dx\right]_0^T \\ &+ \underbrace{\alpha \int_0^T xuv\Big|_0^L dt}_{=0} - \alpha \int_0^T \int_0^L uv \, dxdt + \int_0^T \int_0^L \xi_1(x)g_1(u_t)xu_x + \xi_2(x)g_2(v_t)xv_x \, dxdt. \end{aligned}$$

Now using Young's inequality, and having in mind the energy defined in (2.1), we obtain the main inequality related with the observability for T large enough, namely

$$E(0) \leq C_1 \left(\int_0^T |u_x|^2(L) + |\nabla v|^2(L) + \int_0^T \int_0^L \xi_1^2(x)g_1^2(u_t) + \xi_2^2(x)g_2^2(v_t) \, dxdt \right) dt.$$

where C_1 is a positive constant which do not depend on T.

On the other hand, let the function $\eta \in C^2[0, T]$, with ε being a sufficiently small positive quantity, be such that

$$0 \leq \eta \leq 1, \ \eta(t) = 1 \text{ in } [0, \varepsilon] \cup [T - \varepsilon, T], \ \eta(t) = 0 \text{ in } [2\varepsilon, T - 2\varepsilon].$$

Multiplying the first equation of (1.1) by $xu_x \, \eta$ and performing integration by parts

$$\begin{aligned} \frac{L}{2}\int_0^T |u_x|^2(L)\eta \, dt &= \int_0^T \int_0^L \frac{1}{2}u_t^2\eta - u_t x u_x \eta' + \frac{1}{2}|u_x|^2\eta + \alpha v x u_x \eta \, dxdt \\ &+ \left[\int_0^L u_t u_x x \eta \, dx\right]_0^T + \int_0^T \int_0^L \xi_1(x)g_1(u_t)xu_x\eta \, dxdt. \end{aligned}$$

The equation above, and by definition of η and assertion **H1**, it follows that

$$\begin{aligned} \frac{L}{2}\int_0^\varepsilon |u_x|^2(L) \, dt + \frac{L}{2}\int_{T-\varepsilon}^T |u_x|^2(L) \, dt &= \frac{L}{2}\int_0^\varepsilon |u_x|^2(L)\eta \, dt + \frac{L}{2}\int_{T-\varepsilon}^T |u_x|^2(L)\eta \, dt \\ &\leq \frac{L}{2}\int_0^{2\varepsilon} |u_x|^2(L)\eta \, dt + \frac{L}{2}\int_{T-2\varepsilon}^T |u_x|^2(L)\eta \, dt \\ &\leq \varepsilon C_3[E(0) + E(T)] + C_2[E(0) + E(T)]. \end{aligned}$$

Analogously multiplying the second equation of (1.1) by $x\nabla v\ \eta$ and performing integration by parts

$$\begin{aligned}\frac{L}{2}\int_0^{\varepsilon}|v_x|^2(L)\,dt+\frac{L}{2}\int_{T-\varepsilon}^{T}|v_x|^2(L)\,dt &= \frac{L}{2}\int_0^{\varepsilon}|v_x|^2(L)\eta\,dt+\frac{L}{2}\int_{T-\varepsilon}^{T}|v_x|^2(L)\eta\,dt\\ &\le \frac{L}{2}\int_0^{2\varepsilon}|v_x|^2(L)\eta\,dt+\frac{L}{2}\int_{T-2\varepsilon}^{T}|v_x|^2(L)\eta\,dt\\ &\le \varepsilon C_5[E(0)+E(T)]+C_4[E(0)+E(T)].\end{aligned}$$

where C_i are positive constants which do not depend on T.

From main inequality related with the observability, with the inequalities above, and having in mind the conservation law of energy, we deduce for T large enough(and for a fixed ε), the observability inequality namely

$$E(0)\le C_\varepsilon\left(\int_{\varepsilon}^{T-\varepsilon}|u_x|^2(L)+|v_x|^2(L)\,dt+\int_0^T\int_0^L\xi_1^2(x)g_1^2(u_t)+\xi_2^2(x)g_2^2(v_t)\,dxdt\right)$$

Because of the finite speed of propagation, the observability can take place only if T is large enough, see e.g. [14]. Furthermore, it is important to emphasize that the positive constant C_ε does not depend on the solutions of conservative system. Although we are in 1-D case, we mentioned for the sake of clarity that the region in which the observation applies needs to be large enough to capture all rays of Geometric Optics or the so-called Geometric Control Condition(GCC).

4 Asymptotic Stability

The main purpose of the present section is to determine the asymptotic stability to the damped system (1.1). In order to do that, we will follow the ideas first introduced in [10], and adapted by [11]. For this, let h be defined by

$$\begin{cases} h(x)=h_1(x)+h_2(x),\\ h_1(0)=0=h_2(0),\end{cases}$$

where the h_i are concave, strictly increasing functions such that

$$h_i(sg_i(s))\ge s^2+g_i^2(s),\ \text{ for } |s|\le 1. \tag{4.1}$$

Note that such function can be straightforwardly constructed, given the hypotheses on the functions g_i in Assumption **H1**. With those functions, we define

$$r(\cdot)=h\left(\frac{\cdot}{|Q|}\right) \tag{4.2}$$

where $|Q|:=meas(Q)$, and $Q:=(0,L)\times(0,T)$. As r is monotone increasing, the function $cI+r$ is invertible for all $c\ge 0$. For M a positive constant(the constant M will depend on E(0) and time T_0), we then set

$$p(x)=(cI+r)^{-1}(Mx). \tag{4.3}$$

The function p is easily seen as a positive function , continuous and strictly increasing with $p(0)=0$. Finally, let

$$q(x)=x-(I+p)^{-1}(x). \tag{4.4}$$

The next lemma is proved in [10].

Lemma 4.1. *Let the functions p, q be defined as above. Consider a sequence* (s_n) *of positive numbers which satisfies*

$$s_{m+1} + p(s_{m+1}) \leq s_m.$$

Then $s_m \leq S(m)$ *where S(t) is a solution of the differential equation*

$$\begin{cases} S'(t) + q(S(t)) = 0, \\ S(0) = s_0. \end{cases}$$

Moreover,

$$\lim_{t\to\infty} S(t) = 0,$$

if $p(x) > 0$ *for* $x > 0$.

Now, considering the lemma (2.1), our main task is prove that

$$E(T) \quad \leq \quad C\int_0^T \int_0^L \{\xi_1(x)(u_t^2 + g_1^2(u_t)) + \xi_2(x)(v_t^2 + g_2^2(v_t))\}dxdt, \tag{4.5}$$

for some $C = (T, E(0)) > 0$ and for T sufficiently large, holds for every weak solution to problem (1.1). Assuming that (4.5) takes place, for estimate each tranche of this inequality we define

$$\begin{aligned} \Omega_u &= \{(x,t) \in Q; |u_t(x,t)| > 1\} \text{ and } \Sigma_u = Q \setminus \Gamma_u, \\ \Omega_v &= \{(x,t) \in Q; |v_t(x,t)| > 1\} \text{ and } \Sigma_v = Q \setminus \Gamma_v, \end{aligned}$$

where Γ_u and Γ_v are boundary, and using the Assumptions **H1** and **H2** with Jensen's inequality, we get

$$\begin{aligned} E(T) \quad &\leq \quad \overline{C}\sum_{i=1}^{2}(k_i^{-1} + K_i)\int_Q \xi_1(x)g_1(u_t)u_t + \xi_2(x)g_2(v_t)v_t dxdt \\ &+ \quad \overline{C}|Q|\sum_{i=1}^{2}(1 + ||\xi_i||_\infty)r\bigg(\int_Q \xi_1(x)g_1(u_t)u_t + \xi_2(x)g_2(v_t)v_t dxdt\bigg), \end{aligned}$$

so we are able to define

$$M = \frac{1}{C|Q|\sum_{i=1}^{2}(1 + ||\xi_i||_\infty)} \quad \text{and} \quad c = \frac{\sum_{i=1}^{2}(k_i^{-1} + K_i)}{C|Q|\sum_{i=1}^{2}(1 + ||\xi_i||_\infty)}.$$

Using (2.1) in inequality above, we can still rewrite it as

$$p(E(T)) \leq E(0) - E(T),$$

and proceeding verbatim as considered in [10], there is a time $T_0 > 0$ such that the solution of problem (1.1) satisfies the following decay rate

$$E(t) \leq S\left(\frac{t}{T_0} - 1\right)E(0), \text{ for all } t \geq T_0, \tag{4.6}$$

where

$$S\left(\frac{t}{T_0} - 1\right)E(0) \searrow 0, \text{ with } \ t \to \infty,$$

for energy $E(t)$, where the scalar function $S(t)$ (nonlinear contraction) is the solution of the following ODE:

$$\begin{cases} S'(t) + q(S(t)) = 0, \\ S(0) = E(0) \end{cases}$$

and the function q is defined in (4.4). Effective computations of the decay rates are given in [15], and in our case it suffices to consider (4.3) and (4.4) with the structure of the function h, which is related to the dissipation by the inequality (4.1). Now, we are in a position to state our main result:

Theorem 4.2. *Assume Assumptions* **H1** *and* **H2**. *Then the problem* (1.1) *possesses a unique (weak) solution which satisfies the decay rate estimate given in* (4.7)*, if* $0 < E(0) < M$.

Proof. In order to prove Theorem (4.2) it is sufficient to prove inequality (4.5). Since

$$E(t) \leq E(0), \forall t > 0,$$

it is enough to prove, for all given $T, M > 0$, T sufficiently large, that there is a constant $C = C(T, M) > 0$ such that

$$E(0) \quad \leq \quad C\int_0^T \int_0^L \{\xi_1(x)(u_t^2 + g_1^2(u_t)) + \xi_2(x)(v_t^2 + g_2^2(v_t))\}dxdt, \tag{4.7}$$

happens for every $\{u, v, u_t, v_t\}$ strong solution of (1.1)), checking $0 < E(0) < M$.

Let us suppose that (4.7) is not verified and let

$$W_{0n} = \{u_{0n}, v_{0n}, u_{1n}, v_{1n}\}$$

be a sequence of initial data limited in space $\mathcal{H}$ for the corresponding solution

$$W_n(t) = \{u_n, v_n, u_n', v_n'\}_{n\in\mathbb{N}}$$

of system (1.1). One has $E_n(0) < M$ i.e, uniformly bounded in n, and

$$\lim_{n\to\infty} \frac{E_n(0)}{\int_0^T \int_0^L \xi_1(x)(u_t^2 + g_1^2(u_t)) + \xi_2(x)(v_t^2 + g_2^2(v_t))\, dxdt} = \infty,$$

or equivalently,

$$\lim_{n\to\infty} \frac{\int_0^T \int_0^L \xi_1(x)(u_t^2 + g_1^2(u_t)) + \xi_2(x)(v_t^2 + g_2^2(v_t))\, dxdt}{E_n(0)} = 0. \tag{4.8}$$

Since $(E_n(0))$ is uniformly bounded in n, we obtain

$$u_n' \overset{*}{\rightharpoonup} u' \text{ in } L^\infty(0, T; L^2(0, L)), \tag{4.9}$$

$$v_n' \overset{*}{\rightharpoonup} v' \text{ in } L^\infty(0, T; L^2(0, L)), \tag{4.10}$$

$$\nabla u_n \overset{*}{\rightharpoonup} u_x \text{ in } L^\infty(0, T; L^2(0, L)), \tag{4.11}$$

$$\nabla v_n \overset{*}{\rightharpoonup} v_x \text{ in } L^\infty(0, T; L^2(0, L)), \tag{4.12}$$

From (4.11), and by the Poincaré inequality, we obtain

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0, T; L^2(0, L)), \tag{4.13}$$

and therefore by the Aubin-Lions theorem, we get

$$u_n \to u \text{ strongly in } L^2(0, T; L^2(0, L)). \tag{4.14}$$

Analogously, from Poincaré inequality and by the Aubin-Lions theorem we also have

$$v_n \to v \text{ strongly in } L^2(0, T; L^2(0, L)). \tag{4.15}$$

Since $(E_n(0))$ is bounded, the inequality (4.8) yields

$$\lim_{n\to\infty}\int_0^T\int_0^L \xi_1(x)(u_t^2+g_1^2(u_t))\,dxdt = \ 0 \ = \lim_{n\to\infty}\int_0^T\int_0^L \xi_2(x)(v_t^2+g_2^2(v_t))\,dxdt. \qquad (4.16)$$

Using the assumptions **H1** and **H2** and after (4.16), we obtain

$$\lim_{n\to\infty}\int_0^T\int_{I_\varepsilon} u_n'^2\,dx\,dt = \ 0 \ = \lim_{n\to\infty}\int_0^T\int_{I_\varepsilon} v_n'^2\,dx\,dt.$$

The convergences above and passing to the limit in (1.1), we arrive at

$$\begin{cases} u''-\Delta u+\alpha v=0 \ \text{ in } Q:=(0,L)\times(0,T),\\ v''-\Delta v+\alpha u=0 \ \text{ in } Q,\\ u'=0=v' \text{ in } Q_\varepsilon:=I_\varepsilon\times(0,T). \end{cases} \qquad (4.17)$$

Taking the derivative of (4.17) on t in the distributional sense and substituting $\varphi=u_t$ and $\psi=v_t$, we infer

$$\begin{cases} \varphi_{tt}-\Delta\varphi+\alpha\psi=0 \text{ in } Q,\\ \psi_{tt}-\Delta\psi+\alpha\varphi=0 \text{ in } Q,\\ \varphi=0=\psi \text{ in } Q_\varepsilon. \end{cases}$$

Employing Holmgren's uniqueness theorem we deduce that $\varphi=\psi=0$ in Q, and consequently,

$$u'=v'=0, \ \text{ in } Q. \qquad (4.18)$$

Returning to (4.17) we obtain in Q:

$$\begin{cases} -\Delta u+\alpha v=0 \ \text{ in } Q,\\ -\Delta v+\alpha u=0 \ \text{ in } Q. \end{cases} \qquad (4.19)$$

as $|\alpha|$ is a sufficiently small positive quantity. Multiplying the first equation of (4.19) by u, the second one by v and adding the obtained results, yields

$$\int_Q\{(u_x)^2+(v_x)^2+2\alpha uv\}dxdt=0$$

which implies from norm in $\mathcal{H}$, that $u=0=v$. Now, setting initially

$$w_n:=\sqrt{E_n(0)},\ \overline{\varphi}_n:=\frac{u_n}{w_n}, \ \text{ and } \overline{\psi}_n:=\frac{v_n}{w_n}$$

from (4.8) we obtain

$$0=\lim_{n\to\infty}\int_Q \xi_1(x)\left(\overline{\varphi}_n'^2+\frac{g_1(w_n\overline{\varphi}_n')^2}{w_n^2}\right)+\xi_2(x)\left(\overline{\psi}_n'^2+\frac{g_2(w_n\overline{\psi}_n')^2}{w_n^2}\right)\,dxdt. \qquad (4.20)$$

Furthermore, defining, for each n, the energy $\overline{E}_n(t)$ of the normalized problem as

$$\overline{E}_n(t)=\frac{1}{2}\int_0^L\{\overline{\varphi}_n'^2+\overline{\psi}_n'^2+\nabla\overline{\varphi}_n+\nabla\overline{\psi}_n^2+2\alpha\overline{\varphi}_n\overline{\psi}_n\}\,dx,$$

then,

$$\overline{E}_n(0)=\frac{E_n(0)}{w_n^2}=1, \ \text{ for all } n\in\mathbb{N}. \qquad (4.21)$$

Consequently, making use of the Poincaré inequality and by the Aubin-Lions theorem, one has

$$\overline{\varphi}_n \to \overline{\varphi} \;\text{ in } L^2(0,T;L^2(0,L)),$$
$$\overline{\psi}_n \to \overline{\psi} \;\text{ in } L^2(0,T;L^2(0,L)).$$

Taking into account the convergence above and passing to the limit in the system

$$\begin{cases} \overline{\varphi}_n'' - \Delta\overline{\varphi}_n + \xi_1(x)\dfrac{g_1(w_n\overline{\varphi}_n')}{w_n} + \alpha\overline{\psi}_n = 0, \\ \overline{\psi}_n'' - \Delta\overline{\psi}_n + \xi_2(x)\dfrac{g_2(w_n\overline{\psi}_n')}{w_n} + \alpha\overline{\varphi}_n = 0 \end{cases} \tag{4.22}$$

in Q, we conclude

$$\begin{cases} \overline{\varphi}_n'' - \Delta\overline{\varphi}_n + \alpha\overline{\psi}_n = 0 \;\text{ in } Q, \\ \overline{\psi}_n'' - \Delta\overline{\psi}_n + \alpha\overline{\varphi}_n = 0 \;\text{ in } Q, \\ \overline{\varphi}_{nt} = 0 = \overline{\psi}_{nt} \text{ in } Q_\varepsilon. \end{cases}$$

Similarly, as previously seen, we noticed that $\overline{\varphi}_n = \overline{\psi}_n = 0$. In what follows, we shall prove some technical identities that will play an important role when proving the asymptotic stability by contradiction.

Let $\theta \in C_0^\infty(0,T)$ a cut-off function, with ε being a sufficiently small positive quantity, be defined by

$$\left| \begin{array}{l} 0 \le \theta(t) \le 1, \forall t \in (0,T), \\ \theta(t) = 1, \;\text{ in } [\varepsilon, T-\varepsilon], \\ \theta(t) = 0, \;\text{ when } t = 0, T. \end{array} \right.$$

Multiplying the first equation by $u\theta$, we arrive at

$$\int_Q u_t^2\theta + u_t u\theta' \, dxdt = \int_Q \{u_x^2 + \alpha uv + \xi_1(x)g_1(u_t)u\}\theta \, dxdt. \tag{4.23}$$

And multiplying de first equation by $xu_x\theta$, we obtain

$$\begin{aligned} \int_Q u_t u_x x\theta' \, dxdt + \frac{L}{2}\int_0^T {u_x}^2(L)\theta \, dt \;\; &= \;\; \int_Q \{\alpha v u_x + \xi_1(x)g_1(u_t)u_x\}x\theta \, dxdt \\ &+ \;\; \int_Q \frac{1}{2}(u_t^2 + (u_x)^2)\theta \, dxdt. \end{aligned} \tag{4.24}$$

From (4.23),

$$0 = \int_Q -\overline{\varphi}_n'^2\theta - \overline{\varphi}_n'\overline{\varphi}_n\theta' + (\nabla\overline{\varphi}_n)^2\theta + \alpha\overline{\varphi}_n\overline{\psi}_n\theta + \xi_1(x)\frac{g_1(w_n\overline{\varphi}_n')}{w_n}\overline{\varphi}_n\theta \, dxdt.$$

Taking the weak and strong convergence above, making use of the Poincaré inequality and the Aubin-Lions theorem, and since $\overline{\varphi} = \overline{\psi} = 0$, from the identity (4.20), (4.21) and (4.24) we obtain

$$\lim_{n\to\infty}\int_0^T\int_0^L (\nabla\overline{\varphi}_n)^2\theta \, dxdt = 0. \tag{4.25}$$

By other hand, from (4.24)

$$\begin{aligned} 0 = \;\; & \int_Q \frac{1}{2}\{\overline{\varphi}_n'^2 + (\nabla\overline{\varphi}_n)^2\}\theta - \overline{\varphi}_n'\nabla\overline{\varphi}_n x\theta' \, dxdt - \frac{L}{2}\int_0^T (\nabla\overline{\varphi}_n)^2(L)\theta \, dt \\ + \;\; & \int_Q \left\{\alpha\overline{\psi}_n + \xi_1(x)\frac{g_1(w_n\overline{\varphi}_n')}{w_n}\right\}\nabla\overline{\varphi}_n x\theta \, dxdt. \end{aligned}$$

Considering the above convergence, and taking the identity above into consideration, one has

$$\lim_{n\to\infty} \frac{L}{2}\int_0^T (\nabla\overline{\varphi}_n)^2(L)\theta\, dt = 0. \tag{4.26}$$

Analogously, we obtain

$$\lim_{n\to\infty} \int_0^T \int_0^L (\nabla\overline{\psi}_n)^2\theta\, dxdt = 0, \tag{4.27}$$

and

$$\lim_{n\to\infty} \frac{L}{2}\int_0^T (\nabla\overline{\psi}_n)^2(L)\theta\, dt = 0. \tag{4.28}$$

In order to obtain the desired contradiction let us rewrite (3)) in terms of problem (4.22), that is

$$\overline{E}_n(0) \quad \leq \quad C_\varepsilon \left(\int_\varepsilon^{T-\varepsilon} (\nabla\overline{\varphi}_n)^2(L) + (\nabla\overline{\psi}_n)^2(L)\, dt + \int_Q \xi_1(x)\frac{g_1(w_n\overline{\varphi}_n')^2}{w_n^2} + \xi_2(x)\frac{g_2(w_n\overline{\psi}_n')^2}{w_n^2}\, dxdt \right).$$

Combining (4.21), (4.20), (4.26), (4.28) and the inequality above into consideration we obtain that $1 \leq 0$, which is a contradiction, proving inequality (4.7) as we desired. □

5 Conclusion

This model describes the evolution of a system of two elastic membranes. We generalizes the results of [4] and [5] to the case of non-linear dampings acting in both the equations, under suitable hypothesis on the non-linearities, and we able to prove a uniform decay for solutions. Then, the hypotheses (both the equation are stabilized, the supports of the dampings are the same set, which is of non-zero measure) seem to me overmuch, and we wonder if one could prove the same result under more general assumptions. Moreover, both waves have the same propagation speed, but maybe it would be possible also to consider the case of equations with variable coefficients. Finally, one could also look at how the regularity of the coefficients affects the stabilization properties, in the same spirit of recent papers [7], [8], and [6], about observability and controllability of wave equations in dimension 1.

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Author has declared that no competing interests exist.

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