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An Exposition of the Eight Basic Measures in Diagnostic Testing Using Several Pedagogical Tools

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Authors' contributions

This work was carried out in collaboration between the two authors. Author AMAR designed the study, performed the analysis, solved the examples and wrote the manuscript. Author FAT managed the literature search and drew the figures. Both authors read and approved the final manuscript.

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Abstract

Diagnostic testing concerning categorical or dichotomized variables is ubiquitous in many fields including, in particular, the field of clinical or epidemiological testing. Typically, results are aggregated in two-by-two contingency-table format, from which a surprisingly huge number of indicators or measures are obtained. In this paper, we study the eight most prominent such measures, using their medical context. Each of these measures is given as a conditional probability as well as a quotient of certain natural frequencies. Despite its fundamental theoretical importance, the conditional-probability interpretation does not seem to be appealing to medical students and practitioners. This paper attempts a partial remedy of this situation by visually representing conditional probability formulas first in terms of two-variable Karnaugh maps and later in terms of simplified acyclic (Mason) Signal Flow Graph (SFGs), resembling those used in digital communications or DNA replication. These graphs can be used, among other things, as parallels to trinomial graphs that function as a generative model for the ternary problems of conditional probabilities, which were earlier envisioned by Pedro Huerta and coworkers. The arithmetic or algebraic reading or solving of a typical conditional-probability problem is facilitated and guided by embedding the problem on the SFG that parallels a trinomial graph. Potential extensions of this work include utilization of more powerful features of SFGs, interrelations with Bayesian Networks, and reformulation *via* Boolean-based probability methods.

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Keywords: Diagnostic testing; conditional probability; natural frequency; signal flow graph; trinomial graph; epidemiology; Boolean-based probability.

1 Introduction

A ubiquitous tool of scientific analysis and research is the two-by-two contingency matrix (known also by a variety of other names such as the confusion table, the frequency matrix or the agreement table [1–8]. This table arises in many diverse applications such as clinical testing, criminal investigations, judicial trials, lie detection, null-hypothesis acceptance/rejection, quality control, industrial management, satellite mapping, digital communications, DNA identification, forensic reasoning and machine classification (albeit our current treatment will be mainly restricted, for the sake of clarity, to the context of clinical testing). Generally, the table is a means of classification of data describing a discrete sample of an arbitrary kind in which each individual case of the sample either possesses or does not possess a certain attribute, trait, or condition to be detected. This attribute is possibly a categorical binary variable (sick/healthy, guilty/innocent, false/true, black/white, …, *etc*.) or a continuous variable to be dichotomized using a specific threshold. A certain test, operator or metric j (called a reference) partitions the sample into two groups, one with the attribute and another without it. A second test, operator or metric i (called the assessed metric) introduces its own partitioning of the sample, again into two groups. Therefore, each individual case among the population sample must fall into one of four categories. The total numbers of cases within these categories are entered into the four cells of the contingency table or matrix [1-8].

This paper is a tutorial exposition of the eight most prominent measures or indicators used in diagnostic testing. These are the Sensitivity or True Positive Rate (TPR), the Specificity or True Negative Rate (TNR), the Positive and Negative Predictive Values (PPV and NPV), together with their respective complements, namely the False Negative Rate (FNR), False Positive Rate (FPR), False Discovery rate (FDR) and False Omission Rate (FOR) [8]. The paper uses four distinct ways for interpreting these measures in terms of

1. Conditional Probabilities, which allow fundamental theoretical interrelations among these measures via the Total probability Theorem and Bayes' Theorem [8–14].
2. Quotients of natural frequencies, which provide a readily comprehendible format for medical students and practitioners [8,15–23].
3. Signal flow graphs, which are very convenient in representing linear relations [24–30], including, in particular, the aforementioned Total Probability Theorem [31–41].
4. Trinomial graphs, which are special graphs designed by Pedro Huerta and coworkers [42–48] to represent a special hind of conditional-probability problems called ternary problems. These Trinomial graphs are slightly enhanced herein to stress their parallelism with signal flow graphs.

The organization of the rest of this paper is a follows. Section 2 is a detailed exposition of the 2×2 contingency table and the eight most prominent measures used in association with this table. Mathematical formulas and visual explanation for these measures are given and their natural-frequency and conditional-probability interpretation are interrelated pictorially via Karnaugh map. Section 3 introduces Signal Flow Graphs (SFG), and uses them to inter-relate the eight aforementioned measures. Section 4 shows that our SFGs can be used to mimic trinomial graphs used in the study of Ternary problems of conditional probability. Section 5 concludes the paper.

2 The Contingency Table

In this section, we reproduce from Rushdi & Rushdi [8] a quick review of the concept and structure of a contingency matrix (also known as a frequency table or as a confusion matrix). Fig. 1 demonstrates a two-by-two contingency matrix for metric or test i with respect to metric or test j. Two dichotomous variables are involved, each of which is of a value belonging to the set $\{+1,-1\}$ of indices. The metric or test i (typically a new test to be assessed) is reporting positive cases (of the value + 1), in which the (usually adverse) attribute, trait, or condition is present, or reporting negative cases (of the value - 1), in which this condition is absent. This test or metric is judged or evaluated by a reference or standard metric j (typically a gold standard or the best test available), which has its own labeling of cases, again as positive or negative. If the reference metric j agrees with the assessed metric i then j designates the case of i as "true", and if j disagrees with i then the reference metric designates the case of the assessed one as "false". The sum of the four element $TP_{ij}, FP_{ij}, FN_{ij}$, and TN_{ij} in the two-by-two contingency matrix of Fig. 1 is the size of the reported population or total number of cases N. Each of these elements can be expressed in terms of probabilities of intersection of events concerning the metric i and j namely.

$$TP_{ij} = N * P\big((i = +1) \cap (j = +1)\big) \tag{1}$$

$$FP_{ij} = N * P\big((i = +1) \cap (j = -1)\big) \tag{2}$$

$$FN_{ij} = N * P\big((i = -1) \cap (j = +1)\big) \tag{3}$$

$$TN_{ij} = N * P\big((i = -1) \cap (j = -1)\big) \tag{4}$$

As Rushdi & Rushdi [8] point out, the above four elements of the contingency matrix represent non-normalized conjunctive probabilities. Adding elements in the same column yields the two complementary (non-normalized) probabilities concerning the reference metric j namely.

$$TP_{ij} + FN_{ij} = N * P(j = +1) \tag{5}$$

$$FP_{ij} + TN_{ij} = N * P(j = -1) \tag{6}$$

Likewise, adding elements in the same row yields the two complementary probabilities concerning the assessed metric i, namely.

$$TP_{ij} + FP_{ij} = N * P(i = +1) \tag{7}$$

$$FN_{ij} + TN_{ij} = N * P(i = -1) \tag{8}$$

Table 1 provides a brief mathematical listing of the eight most prominent measures or indicators commonly used in diagnostic medicine, expresses each of them in terms of the elements of the contingency matrix, and interpreters each of them in terms of conditional probabilities and natural frequencies. Table 2 is an extended version of Table 1, in which further visual interpretation is provided. Since each of the eight measures is a probability, it is a dimensionless number of a real value that belongs to the unit interval $[0.0,1.0]$.The four measures in the L.H.S. of Table 2 are direct or agreement measures while the four measures in the R.H.S. are their complements (to 1) and hence serve as discrepancy or disagreement measures between the two tests or metrics i and j. For each of the eight measures, we reproduce the contingency table as a Karnaugh map (following Rushdi & Rushdi [8]). Each of the measures ia a conditional probability of the form

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0 \tag{9}$$

We supplement the Karnaugh map for each measure by

(a) a light red loop depicting the probability of the conditioned event $P(A)$.
(b) a bold blue loop representing the probability of the conditioning event $P(B)$. This loop replaces the full matrix rectangle as the sample space, or as the certain event.
(c) a bold black loop representing the intersection of the two aforementioned loops or the probability $P(A \cap B)$ of the two corresponding events. Hence the considered measure is the conditional probability

$P(A|B)$ given by the area of the black loop divided by that of the blue loop. This allows rewriting the measure as a quotient of a certain natural frequency (entry within the black loop) by the sum of two natural frequencies (the two entries within the blue loop). The aforementioned Karnaugh map representations in Table 2 are essentially similar to those of Venn diagram (albeit with rectilinear rather than curvilinear boundaries). The visual definitions in Table 2 are hopefully easier to comprehend and remember and provide a straightforward link reconciling the conceptual difference between the conditional probability and natural frequency definitions.

3 Signal Flow Graph Interpretation

A linear signal flow graph (SFG), is a specialized directed graph in which nodes represent system variables, and branches or edges represent transmittance or functional connections between pairs of nodes. An edge emanating from a certain node and incident on a node brings to the latter node the value of the former node weighted (multiplied) by the transmittance carried by the edge. There are two main closely-related types of an SFG [27–28], namely

- Mason SFG (employed herein) [24–25]: This is an SFG in which the weighted sum of nodes having arrows incident on specified node (sum of the values of the nodes, each multiplied by the transmittance on its edge towards the specified node) is equal to the value of the specified node.
- Coates SFG [26]: This is an SFG in which the aforementioned weighted sum of nodes with arrows incident on specified node is equal to zero.

Therefore, a linear signal flow graph is associated with a set of linear scalar equations, or, equivalently, a matrix equation. The SFG can be used to solve n linearly-independent scalar equations in n unknowns (i.e., an equation involving a full-rank matrix).

Following a suggestion by Rushdi & Rushdi [8], we add the Mason SFG to our arsenal of methods for interpreting, representing, and comprehending the eight diagnostic measures of Section 2. Using the jargon of digital communication and DNA replication [38], we introduce in Fig. 2, what we call a Channel Diagram and an Inverse Channel Diagram for interpreting the 8 most prominent diagnostic measures. A word of warning is necessary here. The Channel Diagram and the Inverse Channel Diagram should be used one at a time and should not be used together. If one diagram is superimposed on the other, the resulting SFG will be singular, i.e., its Δ will be zero leading to a zero in the denominator of any Mason gain formula deduced from the SFG.

Figs. 3-8 to follow, give alternative pedagogically useful variants for the basic idea in Fig. 2. Fig. 3 is an expanded Channel Diagram with $P(j = -1)$ and $P(j = +1)$ as sources, while Fig. 4. is an expanded Channel Diagram with conditional probabilities conditioned on j as sources. Figs. 5 and 6 are inverse versions of Figs. 3 and 4, respectively. Fig. 7 is a combination of the Channel Diagram and the Inverse Channel Diagram in Figs. 3 – 6, where either red arrows or black arrows are applicable at a time. Fig. 8. is a combination of Fig. 4 and Fig. 6, where either black/blue entities (Fig. 4) OR red/blue entities (Fig. 6) should be used. The whole figure is an SFG replica of Fig. 9 in Pedro Huerta [47].

4 Ternary Problems of Conditional Probabilities

A probability problem is a conditional probability problem if in its formulation at least one of the quantities explicitly mentioned (either as known quantity or as the unknown to be found) could be interpreted as a conditional probability. A ternary problem of a conditional probability is a conditional probability problem formulated with exactly three known quantities and a single unknown quantity to be solved for. To classify ternary problems of conditional probability, we paraphrase and enhance the findings of Pedro Huerta and coworkers [42–48] as follows. We introduce the three integer variables of **Known** probabilities
L= level of the problem = the number of known **conditional** probabilities, $0 \leqslant L \leqslant 8$.

C= Category of the problem = the number of known **marginal** probabilities, $0 \leqslant C \leqslant 4$.

J= The number of known **joint** probabilities in a ternary problem, $0 \leqslant J \leqslant 4$.

The total number of these known quantities must be 3 for a ternary problem, i.e.,

$$L + C + J = 3 \tag{10}$$

Equation (10) indicates that the upper value for each of *L*, *C* and *J* should be reset to 3. It also indicates that J is fixed once L and C are specified. Since only 2 of the marginal probabilities are really independent (the other two being their complements), we assume $C \leqslant 2$,i.e., the marginal probabilities known are one or none of $P(i = +1)$ and $P(i = -1)$ plus one or none of $P(i = +1)$ or $P(i = -1)$. Since the number J of joint probabilities in a ternary problem is fixed by equation (10) in terms of the numbers *L* and *C*, we express *J* in Fig. 9 via a multi-value Karnaugh map of the two inputs *L* and *C* (where $0 \leqslant L \leqslant 3, 0 \leqslant C \leqslant 2$). Note that *J* is not mentioned explicitly in [42–48], though its inclusion (initially at least) facilitates conceptual clarity considerably. We now introduce a fourth variable T (called the type of the problem) which is the single unknown quantity, where $T \in$\{c,m,j\} such that *T=c* if the desired unknown is a conditional probability, *T=m* if it is a marginal probability. *T=j* if it is a joint probability. We again represent the possible problem space by a multi-value Karnaugh-map of two input variables *L* and *C* where again *L* is 4-valued level belonging to \{0, 1, 2, 3\} and *C* is a 3-valued category belonging to \{0, 1, 2\}. The map has an output which is the 3-valued type $T \in$\{c,m,j\}. Actually, *T* is allowed any value in the set \{c, m, j\} for five cells of the map. For the three cells at the right bottom corner of the map (\{L, C\} = \{3, 1\},\{3, 2\} or \{2, 2\}), this set is replaced by $\emptyset$ since in these cells the condition $(L + C \leqslant 3)$ (resulting form L+C+J = 3 and $0 \leqslant J$) is violated (see Fig. 9). For the three cells of the first column of the map (*L=0) T=c* to ensure that some conditional probability is involved. For the cell *L=1, C=2*, a value *m* is denied for *T*, since the unknown cannot be a marginal probability in the type of problems considered (all marginal probabilities deducible without involvement of conditional probabilities).

A ternary problem of conditional probability that is formulated in the context of a contingency table involves 16 probabilities:

- Four **marginal** probabilities: The probabilities of two basic events $P(i = +1)$ and $P(j = +1)$ the probabilities of the complementary events $P(i = -1)$ and $P(j = -1)$.
- Four **joint** or **intersection** probabilities: $P((i = +1) \cap (j = +1))$, $P(i = +1) \cap (j = -1)$, $P(i = -1) \cap (j = +1)$, and $P(i = -1) \cap (j = -1)$
- Eight **conditional** probabilities, which are the eight indicators in Tables 1 and 2.

There are 18 ternary relations among these 16 probabilities, which are exhibited in Table 3. These 18 relations among the aforementioned 16 probabilities are represented by what is called a trinomial graph by Pedro Huerta and coworkers [42-48], or equivalently by our composite SFG in Figs. 7 and 8. Though the resemblance between each of these SFGs and a trinomial graph is profound, indeed, there is a subtle difference. The signal flow graphs are well known entities while over than a half-century history. They avoid the introduction of too many nodes by using transmittances. Trinomial graphs avoid using transmittances by substituting extra nodes for them. To make our discussion self-contained, we use Fig. 11 to represent the trinomial graph in [42-48]. However, we enhance it by adding colors and arrows, a way to increase its pedagogical usefulness and also a means to bridge the conceptual gap between it and the SFGs in Figs. 7 and 8.

Pedro Huerta [47] distinguishes two main types of problems on the trinomial graph, namely:

1) Problems that allow or admit **arithmetic reading** in which the unknown can be deduced via entirely arithmetic computations that start with the unknowns, possibly produce some intermediate results, and finally end up with the desired unknown.
2) Problems that warrant **algebraic reading** and can be solved by assigning a value *x* to an unknown node, treating this node as if it were known, and then obtaining the unknown as a function of *x* and

also obtaining the value of some intermediate quantity twice (via two different paths), thereby constructing an equation that can be solved for the unknown *x*, and consequently deciding the desired unknown. To illustrate this point, we use Fig. 12 to replicate a problem from Pedro Huerta [47] in which a ternary problem warranting an algebraic reading is treated. We are given the following three conditional probabilities

$$TPR_{ij} = Sens_{ij} = P(i = +1 \vee j = +1) = 0.967 \quad (11)$$

$$TNR_{ij} = Spec_{ij} = P(i = -1 \vee j = -1) = 0.75 \quad (12)$$

$$PPV_{ij} = P(j = +1 \vee i = +1) = 0.95 \quad (13)$$

and are required to find *P(j=+1)*, which is assigned the value *x*. Here *P(i=+1)* is computed by two different means, yielding on equation that can be solved for *x*, namely

$$P (i = +1) = 0.967\,x + 0.25 (1 - x) = \frac{0.967x}{0.95}$$

so that *x* (or the prevalence *P(i=+1))* is $\frac{0.25}{0.30089}$= 0.831.

Table 1. Brief mathematical definitions of commonly used measures or indicators in diagnostic medicine (Rushdi and Rushdi, [8])

Measure or indicator	Formula	Interpretation as probability or conditional probability
Sensitivity True Positive Rate (TPR), Recall, Probability of Detection	$Sens_{ij} = TP_{ij}/(TP_{ij}+FN_{ij})$	$Sens_{ij} = P(i=+1 \mid j=+1)$
Specificity True Negative Rate (TNR)	$Spec_{ij} = TN_{ij}/(TN_{ij}+FP_{ij})$	$Spec_{ij} = P(i=-1 \mid j=-1)$
Precision Positive Predictive Value (PPV)	$PPV_{ij} = TP_{ij}/(TP_{ij}+FP_{ij})$	$PPV_{ij} = P(j=+1 \mid i=+1)$
Negative Predictive Value (NPV)	$NPV_{ij} = TN_{ij}/(TN_{ij}+FN_{ij})$	$NPV_{ij} = P(j=-1 \mid i=-1)$
False Negative Rate (FNR)	$FNR_{ij} = 1-Sens_{ij} = FN_{ij}/(TP_{ij}+FN_{ij})$	$FNR_{ij} = 1-Sens_{ij} = P(i=-1 \mid j=+1)$
False Positive Rate (FPR) (Fall-Out, False Alarm)	$FPR_{ij} = 1-Spec_{ij} = FP_{ij}/(TN_{ij}+FP_{ij})$	$FPR_{ij} = 1-Spec_{ij} = P(i=+1 \mid j=-1)$
False Discovery Rate (FDR)	$FDR_{ij} = 1-PPV_{ij} = FP_{ij}/(TP_{ij}+FP_{ij})$	$FDR_{ij} = 1-PPV_{ij} = P(j=-1 \mid i=+1)$
False Omission Rate (FOR)	$FOR_{ij} = 1-NPV_{ij} = FN_{ij}/(TN_{ij}+FN_{ij})$	$FOR_{ij} = 1-NPV_{ij} = P(j=+1 \mid i=-1)$

i \ j	+ 1	– 1
+ 1	(True Positives)	(False Positives) (Type I Error)
– 1	(False Negatives) (Type II Error)	(True Negatives)

Fig. 1. The two-by-two Contingency Matrix of test or metric i with respect to test or metric *j* (Typically represents reality as a gold standard while *i* is a measure to be assessed)

Table 2. Extended visual definition of the eight measures in Table 1.

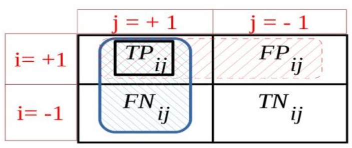
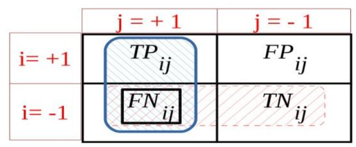
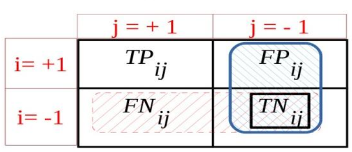
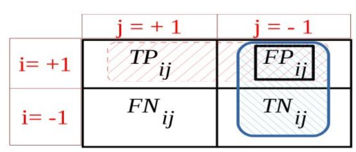
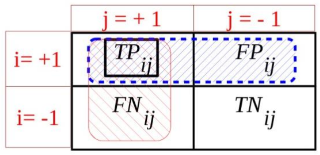
DIRECT MEASURE	COMPLEMENTARY MEASURE
Sensitivity True Positive Rate (TPR), Recall, Probability of Detection	Complement of Sensitivity False Negative Rate (FNR)
	
True Positive Rate (TPR) Natural Frequency: $TPR_{ij} = Sens_{ij} = TP_{ij}/(TP_{ij}+FN_{ij})$ Conditional probability: $TPR_{ij} = Sens_{ij} = P(i=+1 \| j=+1)$	False Negative Rate (FNR) Natural Frequency: $FNR_{ij} = 1-Sens_{ij} = FN_{ij}/(TP_{ij}+FN_{ij})$ Conditional probability: $FNR_{ij} = P(i=-1 \| j=+1)$
Specificity True Negative Rate (TNR)	Complement of Specificity False Positive Rate (FPR)
	
True Negative Rate (TNR) Natural Frequency: $Spec_{ij} = TN_{ij}/(TN_{ij}+FP_{ij})$ Conditional probability: $Spec_{ij} = P(i=-1 \| j=-1)$	False Positive Rate (FPR) Natural Frequency: $FPR_{ij} = 1-Spec_{ij} = FP_{ij}/(TN_{ij}+FP_{ij})$ Conditional probability: $FPR_{ij} = 1-Spec_{ij} = P(i=+1 \| j=-1)$

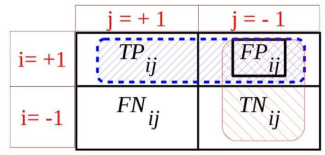
Table 2. (Cont…..)

Precision
Positive Predictive Value (PPV)



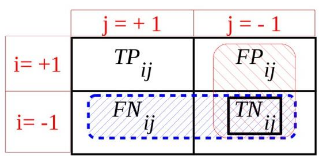
Positive Predictive Value (PPV)	
Natural Frequency	Conditional probability
$PPV_{ij} = TP_{ij}/(TP_{ij}+FP_{ij})$	$PPV_{ij} = P(j=+1 \mid i=+1)$

Complement of Precision
Positive Predictive Value (PPV)



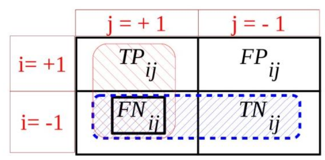
False Discovery Rate (FDR)	
Natural Frequency	Conditional probability
$FDR_{ij} = 1-PPV_{ij} = FP_{ij}/(TP_{ij}+FP_{ij})$	$FDR_{ij} = 1-PPV_{ij} = P(j=-1 \mid i=+1)$

Negative
Predictive Value (NPV)



Negative Predictive Value (NPV)	
Natural Frequency	Conditional probability
$NPV_{ij} = TN_{ij}/(TN_{ij}+FN_{ij})$	$NPV_{ij} = P(j=-1 \mid i=-1)$

Complement of Negative
Predictive Value (NPV)



False Omission Rate (FOR)	
Natural Frequency	Conditional probability
$FOR_{ij} = 1-NPV_{ij} = FN_{ij}/(TN_{ij}+FN_{ij})$	$FOR_{ij} = 1-NPV_{ij} = P(j=+1 \mid i=-1)$

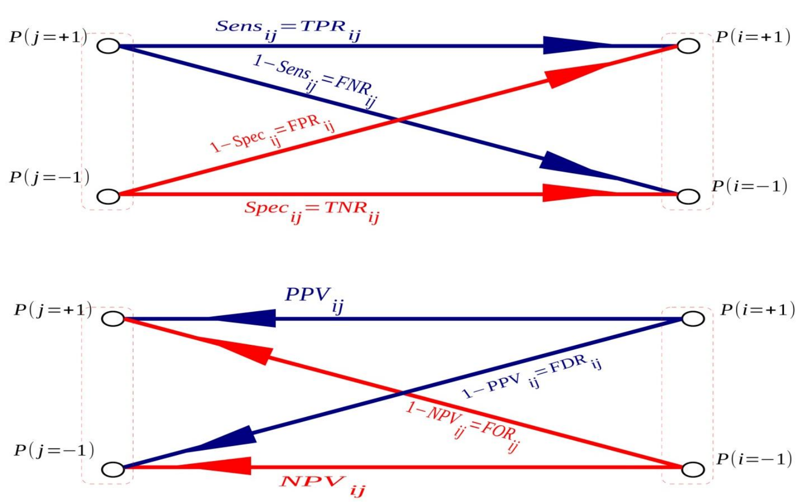


Fig. 2. Channel diagram and inverse channel diagram for the main diagnostic measures

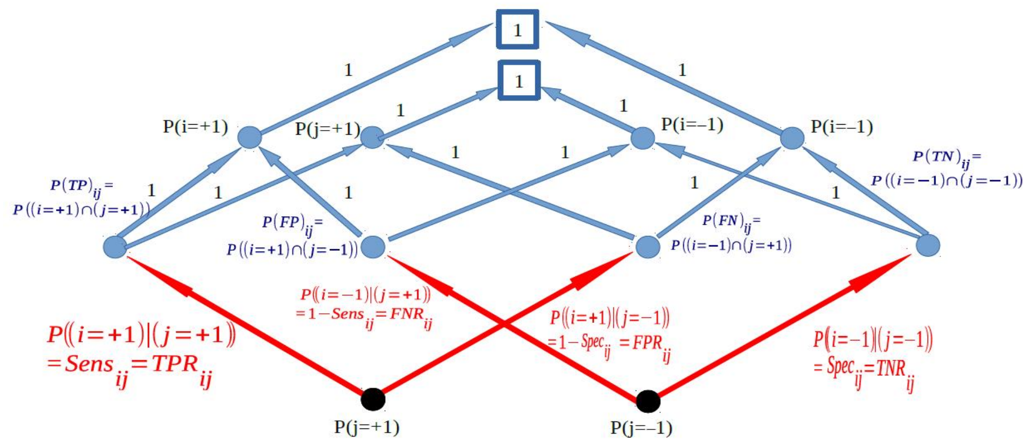


Fig. 3. An expanded Channel Diagram with the true prevalence $P(j = +1)$ and the true non-prevalance $P(j = -1)$as sources

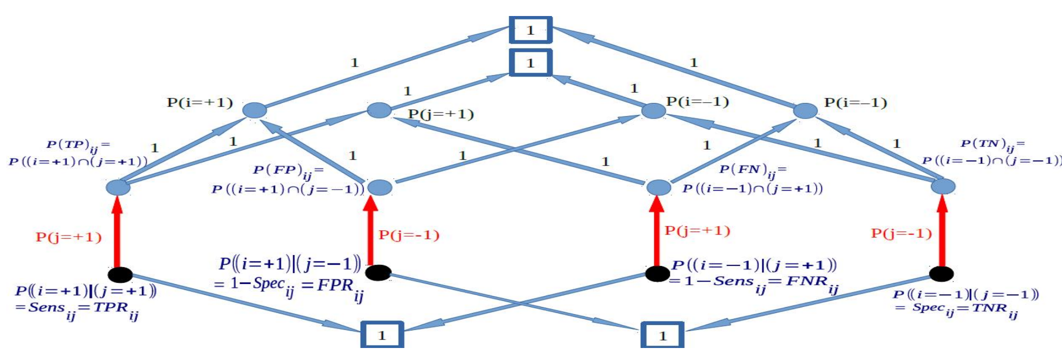


Fig. 4. An expended Channel Diagram with the conditional probabilities that are conditioned on j as sources

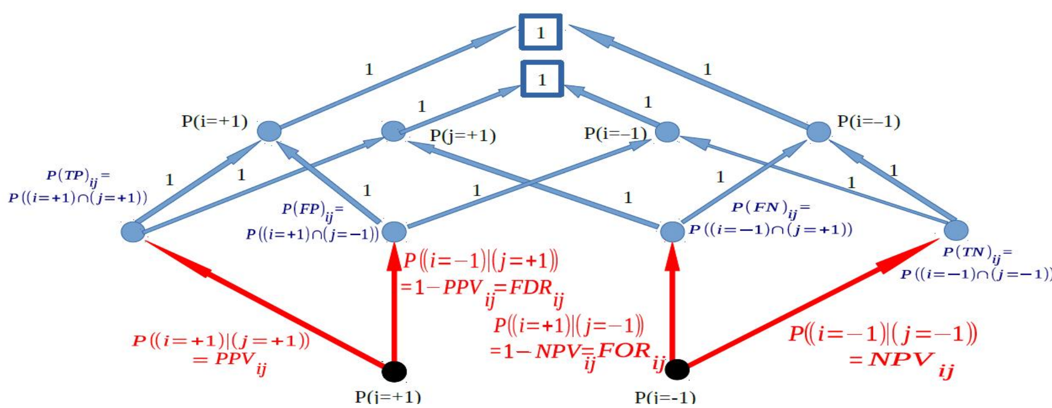


Fig. 5. An expanded Inverse Channel Diagram with $P(j = +1)$ and $P(j = -1)$as sources

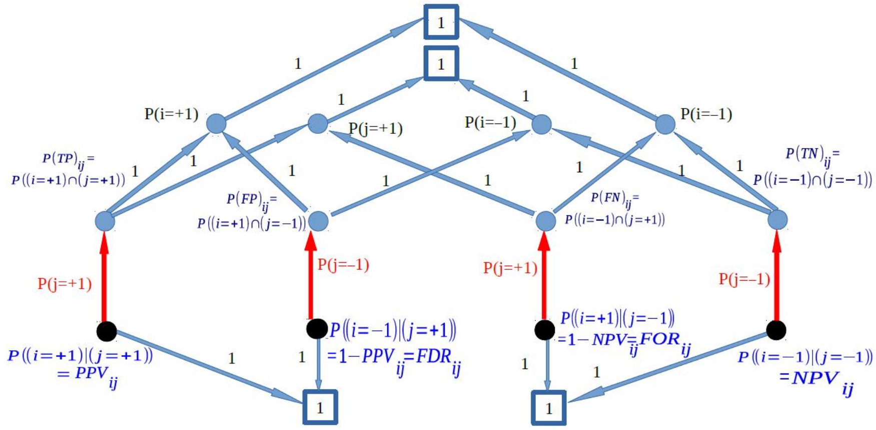


Fig. 6. An expended Inverse Channel Diagram with the conditional probabilities that are conditioned on *i* as sources

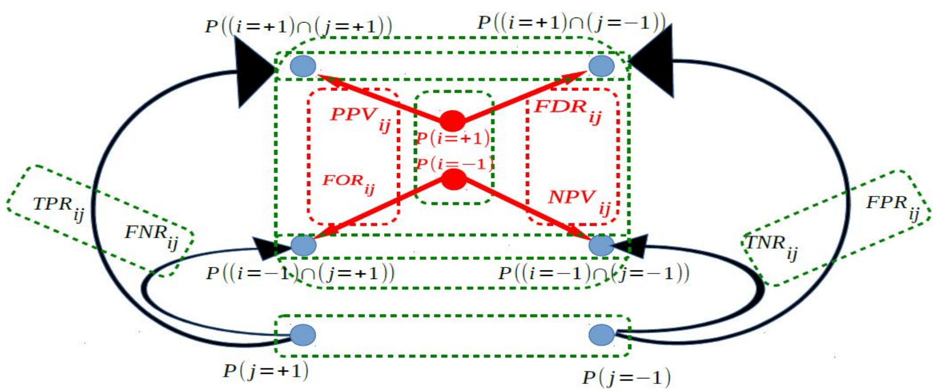


Fig. 7. A combination of the Channel Diagram and Inverse Channel Diagram in Figs. 3 – 6. Either the red arrows or the black arrows are applicable at a time. Nodes that add to 1 are encircled in dotted closed curves

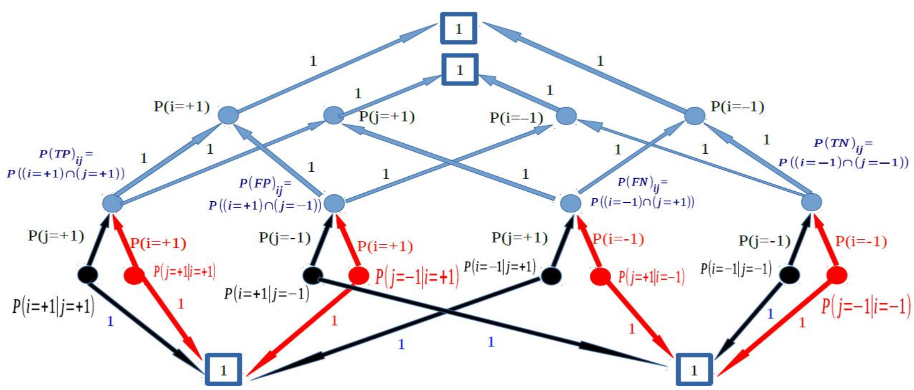


Fig. 8. A combination of Fig. 4 and Fig. 6. Either black/blue entities (Fig. 4) OR red/blue entities (Fig. 6) should be used. The whole figure is an SFG replica of Fig. 9 in Pedro Huerta [47]

C \ L	0	1	2	3
0	3	2	1	0
1	2	1	0	– 1 (rejected)
2	1	0	– 1 (rejected)	– 2 (rejected)

J = 3 - (L + C)

Fig. 9. A multi-value Karnaugh map expressing the number *J* of joint probabilities in a ternary problem in terms of the level *L* and category *C* of the problem.

C \ L	0	1	2	3
0	{c}	{c, m, j}	{c, m, j}	{c, m, j}
1	{c}	{c, m, j}	{c, m, j}	∅
2	{c}	{c, j}	∅	∅

Set to which T belongs

Fig. 10. A multi-value Karnaugh map expressing the type T as function of level Land category C for ternary problems of conditional probability

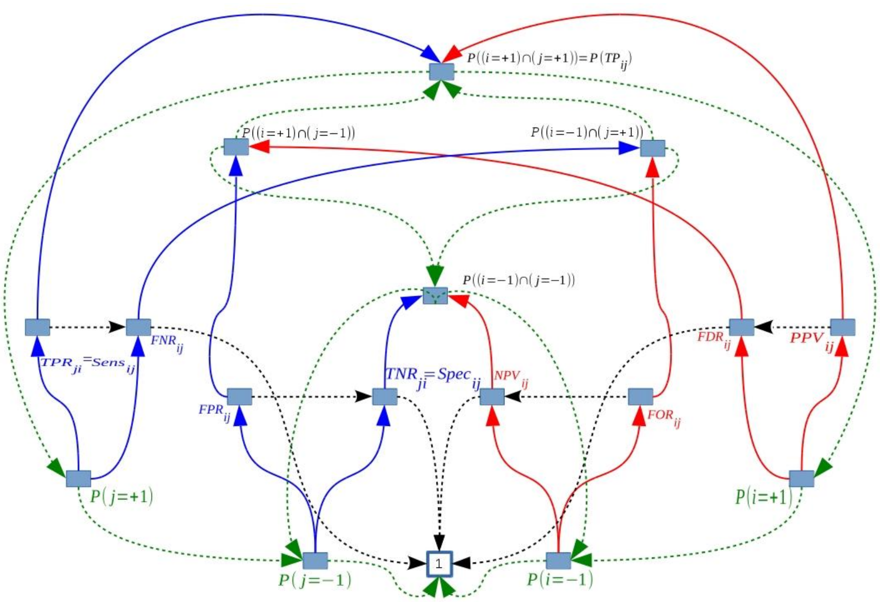


Fig. 11. The trinomial graph of Pedro Huerta [47] enhanced with colors and arrows. Continuous lines indicate multiplication while dotted lines indicate addition.

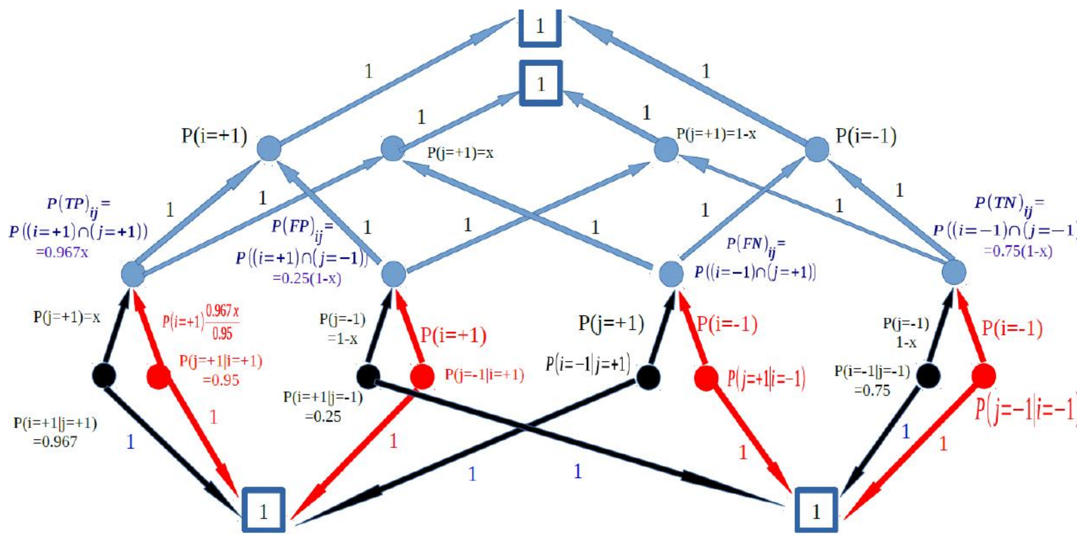


Fig. 12. Algebraic solution of a ternary problem of conditional probability on the SFG of Fig. 8.

Table 3. Then 18 ternary relations of ternary problem of conditional probability in the contingency table contex

Relation Tyre		Relation
Additive Relations (10)	Joint probabilities adding up to a marginal probability (4)	P((i=+1) ∩ (j=+1)) + P((i=+1) ∩ (j=– 1)) = P(i=+1)
		P((i=– 1) ∩ (j=+1)) + P((i=– 1) ∩ (j=– 1)) = P(i=– 1)
		P((i=+1) ∩ (j=+1)) + P((i=-1) ∩ (j=+1)) = P(j=+1)
		P((i=+1) ∩ (j=–1)) + P((i=–1) ∩ (j=– 1)) = P(j=– 1)
	Marginal probabilities adding up to 1 (2)	P(i=+1) + P(i=– 1) = 1
		P(j=+1) + P(j= –1) = 1
	Conditional probabilities adding up to 1 (4)	P((i=+1) | (j=+1)) + P((i=– 1) | (j=+1)) = 1
		P((i=+1) | (j=–1)) + P((i=– 1) | (j=–1)) = 1
		P((j=+1) | (i=+1)) + P((j=– 1) | (i=+1)) = 1
		P((j=+1) | (i=–1)) + P((j=– 1) | (i=–1)) = 1
Multiplicative Relations involving each of the eight conditional probabilities (8)		P(i=+1 | j=+1) P(j=+1) = P((i=+1) ∩ (j=+1))
		P(j=+1 | i=+1) P(i=+1) = P((i=+1) ∩ (j=+1))
		P(i=+1 | j=– 1) P(j=– 1) = P((i=+1) ∩ (j=–1))
		P(j=–1 | i=+1) P(i=+1) = P((i=+1) ∩ (j=–1))
		P(i=–1 | j=+1) P(j=+1) = P((i=–1) ∩ (j=+1))
		P(j=+1 | i=– 1) P(i=– 1) = P((i=–1) ∩ (j=+1))
		P(i=–1 | j=– 1) P(j=–1) = P((i=–1) ∩ (j=–1))
		P(j=–1 | i=–1) P(i=–1) = P((i=–1) ∩ (j=–1))

5 Conclusions

This paper is a tutorial exposition of very important measures that are frequently utilized in widespread applications of diagnostic testing. These measures are viewed herein in the context of their application in epidemiological or clinical testing. We follow Rushdi & Rushdi [8] in using a Karnaugh-map visualization that links the theoretically-appealing definitions of conditional probabilities and the pedagogically popular definitions of natural frequencies.

A notable contribution of the paper is to explore a suggestion of Rushdi & Rushdi [8] for facilitating the treatment of important measures of diagnostic testing. This suggestion is to utilize (Mason) Signal Flow Graphs (SFGs) is relating measures of diagnostic testing. We provide herein several such SFGs. These SFGs can serve as alternate pedagogical aids for the subject. As an offshoot, they are strikingly similar and parallel to trinomial graphs recently developed for the treatment of ternary problems of conditional probability. The paper stresses the similarity between trinomial graphs and a certain class of SFGs by adding color and arrows to the trinomial graphs. This addition constitutes a pedagogical enhancement for trinomial graphs, since it alerts the novice user of these graphs to the fact that the graph edges are directional, saves him/her the trouble of guessing the direction of each edge, and indicates that a node on which two edges of different colors are incident is to be evaluated via one (and only one) of these edges at a time.

The present study is restricted to the eight most prominent measures in the diagnostic testing and also limited to their medical context. Further exploration along the same lines in warranted for other measures or indicators. Definite similarities and potential difference (if any) when other contexts are used have also to be investigated. We have not made a full utilization of the theory of linear SFGs. In particular, we have not encountered any SFG with multiple edges incident simultaneously or a node, or with even a single loop (not to mention several touching or non-touching) loops. Consequently, we made use neither of/the superposition principle nor of the elaborate Mason gain formula. We hope to extend the current work to invoke features and make the most of the powerful SFG theory in its entirety.

Further work is warranted on the relation between the representations used herein and Bayesian Networks (BNs) [49-53], which are directed acyclic graphs in which discrete random variables are assigned to the nodes, together with conditional dependence on parent nodes [49]. Another potentially fruitful direction to explore is the application of concepts of Boolean-based probability [32,40,54-60], which entail the use of switching-algebraic analysis of the indicators of diagnostic testing.

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Competing Interests

Authors have declared that no competing interests exist.

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