**Asian Research Journal of Mathematics**

**Volume 19, Issue 12, Page 69-79, 2023; Article no.ARJOM.110604**
*ISSN: 2456-477X*

# In Constructive and Informal Mathematics, in Contradistinction to any Empirical Science, the Predicate of the Current Knowledge in the Subject is Necessary

## Apoloniusz Tyszka [a*]

[a] *Hugo Kołłątaj University, Balicka 116B, 30-149 Kraków, Poland.*

***Author's contribution***

*The sole author designed, analyzed, interpreted and prepared the manuscript.*

***Article Information***

DOI: 10.9734/ARJOM/2023/v19i12773

**Open Peer Review History:**
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: http://www.sdiarticle5.com/review-history/110604

***Received: 15/10/2023***
***Accepted: 22/12/2023***
***Published: 29/12/2023***

***Original Research Article***

## Abstract

We assume that the current mathematical knowledge $\mathcal{K}$ is a finite set of statements from both formal and constructive mathematics, which is time-dependent and publicly available. Any formal theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. Ignoring $\mathcal{K}$ and its subsets, sets exist formally in $ZFC$ theory although their properties can be time-dependent (when they depend on $\mathcal{K}$) or informal. We explain the distinction between algorithms whose existence is provable in $ZFC$ and constructively defined algorithms which are currently known. By using this distinction, we obtain non-trivially true statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that belong to constructive and informal mathematics and refer to the current mathematical knowledge on $\mathcal{X}$. This and the next sentence justify the article title. For any empirical science, we can identify the current knowledge with that science because truths from the empirical sciences are not necessary truths but working models of truth about particular real phenomena.

**Corresponding author: E-mail: rttyszka@cyf-kr.edu.pl;*

*Keywords:* *Constructive algorithms; constructive mathematics; current knowledge in a scientific discipline; current mathematical knowledge; informal mathematics; known algorithms.*

**2020 Mathematics Subject Classification:** 03A05, 03F65.

# 1 Introduction and Why Such Title of the Article

Let $\mathcal{T}$ denote the set of twin primes. We assume that the current mathematical knowledge $\mathcal{K}$ is a finite set of statements from both formal and constructive mathematics, which is time-dependent and publicly available. Any formal theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. The true statement "*There is no known constructively defined integer* $n$ *such that* $\mathrm{card}(\mathcal{T}) < \omega \Rightarrow \mathcal{T} \subseteq (-\infty, n]$ " is not formal, belongs to $\mathcal{K}$, and may not belong to $\mathcal{K}$ in 2025. The true statement "*There exists a set* $\mathcal{X} \subseteq \{1, \ldots, 49\}$ *such that* $\mathrm{card}(\mathcal{X}) = 6$ *and* $\mathcal{X}$ *never occurred as the winning six numbers in the Polish Lotto lottery*" refers to the current non-mathematical knowledge and does not belong to $\mathcal{K}$. The set $\mathcal{K}$ exists only theoretically. Ignoring $\mathcal{K}$ and its subsets, sets exist formally in $ZFC$ theory although their properties can be time-dependent (when they depend on $\mathcal{K}$) or informal. In every branch of mathematics, the set of all knowable truths is the set of all theorems. This set exists independently of $\mathcal{K}$.

Algorithms always terminate. We explain the distinction between algorithms whose existence is provable in $ZFC$ and constructively defined algorithms which are currently known. By using this distinction, we obtain non-trivially true statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that belong to constructive and informal mathematics and refer to the current mathematical knowledge on $\mathcal{X}$. These results, mainly from [1], and the next sentence justify the article title. For any empirical science, we can identify the current knowledge with that science because truths from the empirical sciences are not necessary truths but working models of truth from a particular context, see [2, p. 610].

The feature of mathematics from the article title is not quite new. Observation 1 is known from the beginning of computability theory and shows that the predicate of the current mathematical knowledge slightly increases the intuitive mathematics.

**Observation 1.** *Church's thesis is based on the fact that the currently known computable functions are recursive, where the notion of a computable function is informal.*

**Observation 2.** *There exists a prime number* $p$ *greater than the largest known prime number.*

In Observation 2, the predicate of the current mathematical knowledge trivially increases the constructive mathematics.

# 2 Basic Definitions and Examples

Algorithms always terminate. Semi-algorithms may not terminate. There is the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in $ZFC$) and *known algorithms* (i.e. algorithms whose definition is constructive and currently known), see [3], [4], [5, p. 9]. A definition of an integer $n$ is called *constructive*, if it provides a known algorithm with no input that returns $n$. Definition 1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven.

**Definition 1.** *We say that a non-negative integer* $k$ *is a known element of* $\mathcal{X}$, *if* $k \in \mathcal{X}$ *and we know an algebraic expression that defines* $k$ *and consists of the following signs:* $1$ *(one),* $+$ *(addition),* $-$ *(subtraction),* $\cdot$ *(multiplication),* $\hat{}$ *(exponentiation with exponent in* $\mathbb{N}$*),* $!$ *(factorial of a non-negative integer),* $($ *(left parenthesis),* $)$ *(right parenthesis).*

The set of known elements of $\mathcal{X}$ is finite and time-dependent, so cannot be defined in the formal language of classical mathematics. Let $t$ denote the largest twin prime that is smaller than $((((((((9!)!)!)!)!)!)!)!)!$. The number $t$ is an unknown element of the set of twin primes.

**Definition 2.** *Conditions* (1)-(5) *concern sets* $\mathcal{X} \subseteq \mathbb{N}$.

(1) *A known algorithm with no input returns an integer* $n$ *satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

(2) *A known algorithm for every* $k \in \mathbb{N}$ *decides whether or not* $k \in \mathcal{X}$.

(3) *There is no known algorithm with no input that returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) = \omega$.

(4) *There are many elements of* $\mathcal{X}$ *and it is conjectured, though so far unproven, that* $\mathcal{X}$ *is infinite.*

(5) $\mathcal{X}$ *is naturally defined. The infiniteness of* $\mathcal{X}$ *is false or unproven.* $\mathcal{X}$ *has the simplest definition among known sets* $\mathcal{Y} \subseteq \mathbb{N}$ *with the same set of known elements.*

Condition (3) implies that no known proof shows the finiteness/infiniteness of $\mathcal{X}$. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions (1)-(4) and is widely known in number theory or naturally defined, where this term has only informal meaning.

Let $[\cdot]$ denote the integer part function.

**Example 1.** *The set* $\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } [\frac{((((((((9!)!)!)!)!)!)!)!)!}{\pi}] \textit{ is odd} \\ \emptyset, & \textit{otherwise} \end{cases}$ *does not satisfy Condition* (3) *because we know an algorithm with no input that computes* $[\frac{((((((((9!)!)!)!)!)!)!)!)!}{\pi}]$. *The set of known elements of* $\mathcal{X}$ *is empty. Hence, Condition* (5) *fails for* $\mathcal{X}$.

**Example 2.** *([3], [4], [5, p. 9]). The function*

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & if\ the\ decimal\ expansion\ of\ \pi\ contains\ n\ consecutive\ zeros \\ 0, & otherwise \end{cases}$$

*is computable because* $h = \mathbb{N} \times \{1\}$ *or there exists* $k \in \mathbb{N}$ *such that*

$$h = (\{0, \ldots, k\} \times \{1\}) \cup (\{k+1,\ k+2,\ k+3,\ \ldots\} \times \{0\})$$

*No known algorithm computes the function* $h$.

**Example 3.** *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & if\ the\ continuum\ hypothesis\ holds \\ \emptyset, & otherwise \end{cases}$$

*is decidable. This* $\mathcal{X}$ *satisfies Conditions* (1) *and* (3) *and does not satisfy Conditions* (2), (4), *and* (5). *These facts will hold forever.*

# 3 Number-Theoretic Results

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2+1$ is infinite, see [6], [7], [8].

**Statement 1.** *Condition* (1) *remains unproven for* $\mathcal{X} = \mathcal{P}_{n^2+1}$.

*Proof.* For every set $\mathcal{X} \subseteq \mathbb{N}$, there exists an algorithm $\mathtt{Alg}(\mathcal{X})$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \mathrm{card}(\mathcal{X}) \in \{0, \omega\} \\ \max(\mathcal{X}), & \text{otherwise} \end{cases}$$

This $n$ satisfies the implication in Condition (1), but the algorithm $\mathtt{Alg}(\mathcal{P}_{n^2+1})$ is unknown because its definition is ineffective. □

**Statement 2.** *The statement*

$$\exists n \in \mathbb{N}\ (\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, n+3])$$

*remains unproven in $ZFC$ and classical logic without the law of excluded middle.*

**Conjecture 1.** *([9, p. 443], [10]). The are infinitely many primes of the form $k!+1$.*

For a non-negative integer $n$, let $\rho(n)$ denote $29.5 + \frac{11!}{3n+1} \cdot \sin(n)$.

**Statement 3.** *The set*

$$\mathcal{X} = \{n \in \mathbb{N} : \textit{the interval } [-1, n] \textit{ contains more than } \rho(n) \textit{ primes of the form } k!+1\}$$

*satisfies Conditions* `(1)-(5)` *except the requirement that $\mathcal{X}$ is naturally defined.* $501893 \in \mathcal{X}$. *Condition* `(1)` *holds with* $n = 501893$. $\mathrm{card}(\mathcal{X} \cap [0, 501893]) = 159827$. $\mathcal{X} \cap [501894, \infty) = \{n \in \mathbb{N} :$ *the interval* $[-1, n]$ *contains at least* $30$ *primes of the form* $k!+1\}$.

*Proof.* For every integer $n \geqslant 11!$, $30$ is the smallest integer greater than $\rho(n)$. By this, if $n \in \mathcal{X} \cap [11!, \infty)$, then $n+1$, $n+2$, $n+3$, $\ldots \in \mathcal{X}$. Hence, Condition `(1)` holds with $n = 11! - 1$. We explicitly know $24$ positive integers $k$ such that $k!+1$ is prime, see [11]. The inequality

$$\mathrm{card}(\{k \in \mathbb{N} \setminus \{0\} : k!+1 \textit{ is prime}\}) > 24$$

remains unproven. Since $24 < 30$, Condition `(3)` holds. The interval $[-1, 11!-1]$ contains exactly three primes of the form $k!+1$: $1!+1$, $2!+1$, $3!+1$. For every integer $n > 503000$, the inequality $\rho(n) > 3$ holds. Therefore, the execution of the following *MuPAD* code

```
m:=0:
for n from 0.0 to 503000.0 do
if n<1!+1 then r:=0 end_if:
if n>=1!+1 and n<2!+1 then r:=1 end_if:
if n>=2!+1 and n<3!+1 then r:=2 end_if:
if n>=3!+1 then r:=3 end_if:
if r>29.5+(11!/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of $\mathcal{X}$. The output ends with the line $[501893.0,\ 159827]$, which proves Condition `(1)` with $n = 501893$ and Condition `(4)` with $\mathrm{card}(\mathcal{X}) \geqslant 159827$. □

T. Nagell proved in [12] (cf. [13, p. 104]) that the equation $x^2 - 17 = y^3$ has exactly $16$ integer solutions, namely $(\pm 3, -2)$, $(\pm 4, -1)$, $(\pm 5, 2)$, $(\pm 9, 4)$, $(\pm 23, 8)$, $(\pm 282, 43)$, $(\pm 375, 52)$, $(\pm 378661, 5234)$. The set

$$\bigcup_{\substack{(x,y) \in \mathbb{Z} \times \mathbb{Z} \\ (x^2 - y^3 - 17) \cdot (y^2 - x^3 - 17) = 0}} \{(x+8)^8\}$$

has exactly $23$ elements. Among them, there are $14$ integers from the interval $[1, 2199894223892]$. Let $\mathcal{W}$ denote the set

$$\bigcup_{\substack{(x,y) \in \mathbb{Z} \times \mathbb{Z} \\ (x^2 - y^3 - 17) \cdot (y^2 - x^3 - 17) = 0}} \{k \in \mathbb{N} :\ k \textit{ is the } (x+8)^8 - th \textit{ element of } \mathcal{P}_{n^2+1}\}$$

From [7], it is known that $\mathrm{card}(\mathcal{P}_{n^2+1} \cap [2,10^{28})) = 2199894223892$ . Hence, $\mathrm{card}(\mathcal{W} \cap [2,10^{28})) = 14$ and 14 elements of $\mathcal{W}$ can be practically computed. The inequality $\mathrm{card}(\mathcal{P}(n^2+1)) \geqslant (378661+8)^8$ remains unproven. The last two sentences and Statement 3 imply the following corollary.

**Corollary 1.** *If we add* $\mathcal{W}$ *to* $\mathcal{X}$ *, then the following statements hold:*

$\mathcal{X}$ *does not satisfy Condition* `(1)`,

$159827 + 14 \leqslant \mathrm{card}(\mathcal{X})$ ,

*the above lower bound is currently the best known,*

$\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathrm{card}(\mathcal{X}) \leqslant 159827 + 23$ ,

*the above upper bound is currently the best known,*

$\mathcal{X}$ *satisfies Conditions* `(2)-(5)` *except the requirement that* $\mathcal{X}$ *is naturally defined.*

*Analogical statements hold, if we add to* $\mathcal{X}$ *the set*

$$\bigcup_{\substack{x \in \mathbb{N} \\ x \text{ divides } 99!}} \{k \in \mathbb{N} : \ k - 501894 \textit{ is the } x-th \textit{ element of } \mathcal{P}_{n^2+1}\}$$

**Definition 3.** *Conditions* `(1a)-(5a)` *concern sets* $\mathcal{X} \subseteq \mathbb{N}$ .

`(1a)` *A known algorithm with no input returns a positive integer* $n$ *satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \ \mathcal{X} \subseteq (-\infty, n]$ .

`(2a)` *A known algorithm for every* $k \in \mathbb{N}$ *decides whether or not* $k \in \mathcal{X}$ .

`(3a)` *There is no known algorithm with no input that returns the logical value of the statement* $\mathrm{card}(\mathcal{X}) < \omega$ .

`(4a)` *There are many elements of* $\mathcal{X}$ *and it is conjectured, though so far unproven, that* $\mathcal{X}$ *is finite.*

`(5a)` $\mathcal{X}$ *is naturally defined. The finiteness of* $\mathcal{X}$ *is false or unproven.* $\mathcal{X}$ *has the simplest definition among known sets* $\mathcal{Y} \subseteq \mathbb{N}$ *with the same set of known elements.*

**Statement 4.** *The set*

$$\mathcal{X} = \Big\{ n \in \mathbb{N} : \textit{ the interval } [-1,n] \textit{ contains more than}$$

$$6.5 + \frac{10^6}{3n+1} \cdot \sin(n) \textit{ squares of the form } k!+1 \Big\}$$

*satisfies Conditions* `(1a)-(5a)` *except the requirement that* $\mathcal{X}$ *is naturally defined.* $95151 \in \mathcal{X}$ . *Condition* `(1a)` *holds with* $n = 95151$ . $\mathrm{card}(\mathcal{X} \cap [0, 95151]) = 30311$ . $\mathcal{X} \cap [95152, \infty) = \{n \in \mathbb{N} :$ *the interval* $[-1,n]$ *contains at least* 7 *squares of the form* $k!+1\}$ .

*Proof.* For every integer $n > 10^6$ , 7 is the smallest integer greater than $6.5 + \frac{10^6}{3n+1} \cdot \sin(n)$ . By this, if $n \in \mathcal{X} \cap (10^6, \infty)$ , then $n+1, \ n+2, \ n+3, \ \ldots \in \mathcal{X}$ . Hence, Condition `(1a)` holds with $n = 10^6$ . It is conjectured that $k!+1$ is a square only for $k \in \{4,5,7\}$ , see [14, p. 297]. Hence, the inequality $\mathrm{card}(\{k \in \mathbb{N} \setminus \{0\} : k!+1 \textit{ is a square}\}) > 3$ remains unproven. Since $3 < 7$ , Condition `(3a)` holds. The interval $[-1, 10^6]$ contains exactly three squares of the form $k!+1$ : $4!+1$ , $5!+1$ , $7!+1$ . Therefore, the execution of the following *MuPAD* code

```
m:=0:
for n from 0.0 to 1000000.0 do
if n<25 then r:=0 end_if:
if n>=25 and n<121 then r:=1 end_if:
if n>=121 and n<5041 then r:=2 end_if:
if n>=5041 then r:=3 end_if:
if r>6.5+(1000000/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of $\mathcal{X}$. The output ends with the line $[95151.0,\ 30311]$, which proves Condition (1a) with $n = 95151$ and Condition (4a) with $\mathrm{card}(\mathcal{X}) \geqslant 30311$. □

**Statement 5.** *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : \mathrm{card}([-1, k] \cap \mathcal{P}_{n^2+1}) < 10^{10000}\}$$

*satisfies the conjunction*

$$\neg(\textit{Condition } \texttt{1a}) \wedge (\textit{Condition } \texttt{2a}) \wedge (\textit{Condition } \texttt{3a}) \wedge (\textit{Condition } \texttt{4a}) \wedge (\textit{Condition } \texttt{5a})$$

For a non-negative integer $n$, let $\theta(n)$ denote the largest integer divisor of $10^{10^{10}}$ smaller than $n$. Let $\kappa : \mathbb{N} \to \mathbb{N}$ be defined by setting $\kappa(n)$ to be the exponent of 2 in the prime factorization of $n + 1$.

**Statement 6.** *([1, p. 250]). The set*

$$\mathcal{X} = \{n \in \mathbb{N} : (\theta(n) + \kappa(n))^2 + 1 \textit{ is prime}\}$$

*satisfies Conditions* (1)-(5) *except the requirement that* $\mathcal{X}$ *is naturally defined. Condition* (1) *holds with* $n = 10^{10^{10}}$.

**Statement 7.** *There exists a naturally defined set* $\mathcal{C} \subseteq \mathbb{N}$ *which satisfies the following conditions* (6)-(11).

(6) *A known and simple algorithm for every* $k \in \mathbb{N}$ *decides whether or not* $k \in \mathcal{C}$.

(7) *There is no known algorithm with no input that returns the logical value of the statement* $\mathrm{card}(\mathcal{C}) = \omega$.

(8) *There is no known algorithm with no input that returns the logical value of the statement* $\mathrm{card}(\mathbb{N} \setminus \mathcal{C}) = \omega$.

(9) *It is conjectured, though so far unproven, that* $\mathcal{C}$ *is infinite.*

(10) *There is no known algorithm with no input that returns an integer* $n$ *satisfying* $\mathrm{card}(\mathcal{C}) < \omega \Rightarrow \mathcal{C} \subseteq (-\infty, n]$.

(11) *There is no known algorithm with no input that returns an integer* $m$ *satisfying* $\mathrm{card}(\mathbb{N} \setminus \mathcal{C}) < \omega \Rightarrow \mathbb{N} \setminus \mathcal{C} \subseteq (-\infty, m]$.

*Proof.* Conditions (6)-(11) hold for

$$\mathcal{C} = \{k \in \mathbb{N} :\ 2^{2^k} + 1 \textit{ is composite}\}$$

It follows from the following three observations. It is an open problem whether or not there are infinitely many composite numbers of the form $2^{2^k}+1$, see [15, p. 159] and [16, p. 74]. It is an open problem whether or not there are infinitely many prime numbers of the form $2^{2^k}+1$, see [15, p. 158] and [16, p. 74]. Most mathematicians believe that $2^{2^k}+1$ is composite for every integer $k \geqslant 5$, see [17, p. 23]. □

# 4 A Consequence of the Physical Limits of Computation

**Statement 8.** *No set* $\mathcal{X} \subseteq \mathbb{N}$ *will satisfy Conditions* `(1)-(4)` *forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in* 1 *second or less.*

*Proof.* The proof goes by contradiction. We fix an integer $n$ that satisfies Condition `(1)`. Since Conditions `(1)-(3)` will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n+1 \notin \mathcal{X},\ n+2 \notin \mathcal{X},\ n+3 \notin \mathcal{X},\ \dots \qquad \texttt{(T)}$$

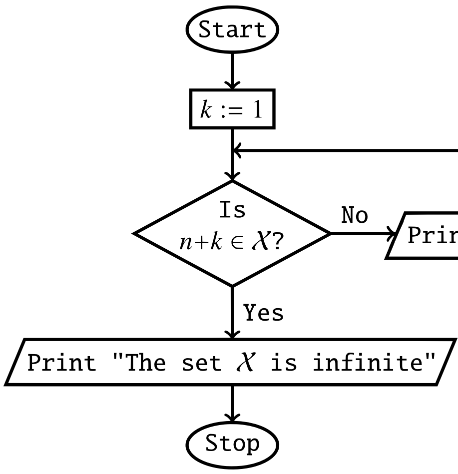


**Fig. 1. Semi-algorithm that terminates if and only if $\mathcal{X}$ is infinite**

The sentences from the sequence `(T)` and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $\mathcal{X}$ is finite, contrary to the conjecture in Condition `(4)`. □

The physical limits of computation ([18]) disprove the assumption of Statement 8.

# 5 Satisfiable Conjunctions which Consist of Conditions `(1)-(5)` and their Negations

**Open Problem 1.** *Is there a set* $\mathcal{X} \subseteq \mathbb{N}$ *which satisfies Conditions* `(1)-(5)`*?*

Open Problem 1 asks about the existence of a year $t \geqslant 2023$ in which the conjunction

$$(Condition\ \texttt{1}) \land (Condition\ \texttt{2}) \land (Condition\ \texttt{3}) \land (Condition\ \texttt{4}) \land (Condition\ \texttt{5})$$

will hold for some $\mathcal{X} \subseteq \mathbb{N}$. For every year $t \geqslant 2023$ and for every $i \in \{1,2,3\}$, a positive solution to Open Problem $i$ in the year $t$ may change in the future. Currently, the answers to Open Problems 1–5 are negative.

The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(Condition\ 1) \wedge (Condition\ 2) \wedge (Condition\ 3) \wedge (Condition\ 4) \wedge (Condition\ 5)$$

The set $\mathcal{X} = \{0,\dots,10^6\} \cup \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(Condition\ 1) \wedge (Condition\ 2) \wedge (Condition\ 3) \wedge (Condition\ 4) \wedge \neg(Condition\ 5)$$

Let $f(1) = 10^6$, and let $f(n+1) = f(n)^{f(n)}$ for every positive integer $n$. The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \ if \ 2^{2^{f(9^9)}} + 1 \ is \ composite \\ \{0,\dots,10^6\}, \ otherwise \end{cases}$$

satisfies the conjunction

$$(Condition\ 1) \wedge (Condition\ 2) \wedge \neg(Condition\ 3) \wedge (Condition\ 4) \wedge \neg(Condition\ 5)$$

**Open Problem 2.** *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$(Condition\ 1) \wedge (Condition\ 2) \wedge \neg(Condition\ 3) \wedge (Condition\ 4) \wedge (Condition\ 5)?$$

The numbers $2^{2^k} + 1$ are prime for $k \in \{0,1,2,3,4\}$. It is open whether or not there are infinitely many primes of the form $2^{2^k} + 1$, see [15, p. 158] and [16, p. 74]. It is open whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [15, p. 159] and [16, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geqslant 5$, see [17, p. 23]. The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \ if \ 2^{2^{f(9^9)}} + 1 \ is \ composite \\ \{0,\dots,10^6\} \cup \\ \{n \in \mathbb{N} : n \ is \ the \ sixth \ prime \ number \ of \ the \ form \ 2^{2^k} + 1\}, \ otherwise \end{cases}$$

satisfies the conjunction

$$\neg(Condition\ 1) \wedge (Condition\ 2) \wedge \neg(Condition\ 3) \wedge (Condition\ 4) \wedge \neg(Condition\ 5)$$

**Open Problem 3.** *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$\neg(Condition\ 1) \wedge (Condition\ 2) \wedge \neg(Condition\ 3) \wedge (Condition\ 4) \wedge (Condition\ 5)?$$

It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \mathcal{P}_{n^2+1}$ will solve Open Problem 2. The same is true for Open Problem 3. It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \{k \in \mathbb{N} : 2^{2^k} + 1 \ is \ composite\}$ will solve Open Problem 1. The same is true for Open Problems 2 and 3.

Table 1 shows satisfiable conjunctions of the form

$$\#(\text{Condition } 1) \wedge (\text{Condition } 2) \wedge \#(\text{Condition } 3) \wedge (\text{Condition } 4) \wedge \#(\text{Condition } 5)$$

where $\#$ denotes the negation $\neg$ or the absence of any symbol.

**Table 1. Five satisfiable conjunctions**

| | (Cond. 2) $\wedge$ (Cond. 3) $\wedge$ (Cond. 4) | (Cond. 2) $\wedge$ $\neg$(Cond. 3) $\wedge$ (Cond. 4) |
|---|---|---|
| (Cond. 1) $\wedge$ (Cond. 5) | Open Problem 1 | Open Problem 2 |
| (Cond. 1) $\wedge$ $\neg$(Cond. 5) | $\mathcal{X} = \{n \in \mathbb{N}:$ *the interval* $[-1, n]$ *contains more than* $29.5 + \frac{11!}{3n+1} \cdot \sin(n)$ *primes of the form* $k! + 1\}$ | $\mathcal{X} = \begin{cases} \mathbb{N}, \textit{ if } 2^{2^{f(9^9)}} + 1 \textit{ is composite} \\ \{0, \ldots, 10^6\}, \textit{ otherwise} \end{cases}$ |
| $\neg$(Cond. 1) $\wedge$ (Cond. 5) | $\mathcal{X} = \mathcal{P}_{n^2+1}$ | Open Problem 3 |
| $\neg$(Cond. 1) $\wedge$ $\neg$(Cond. 5) | $\mathcal{X} = \{0, \ldots, 10^6\} \cup \mathcal{P}_{n^2+1}$ | $\mathcal{X} = \begin{cases} \mathbb{N}, \textit{ if } 2^{2^{f(9^9)}} + 1 \textit{ is composite} \\ \{0, \ldots, 10^6\} \cup \{n \in \mathbb{N} : n \textit{ is the sixth prime number of the form } 2^{2^k} + 1\}, \textit{ otherwise} \end{cases}$ |

# 6 Subsets of $\mathbb{N}$ and their Threshold Numbers

**Definition 4.** *We say that an integer* $n$ *is a threshold number of a set* $\mathcal{X} \subseteq \mathbb{N}$, *if* $\text{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer $n$ is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

**Example 4.** *The set*

$$\mathcal{X} = \{k \in \mathbb{N}: \textit{ any proof in ZFC of length } k \textit{ or less does not show that } \emptyset \neq \emptyset\}$$

*conjecturally equals* $\mathbb{N}$. *No effectively computable integer* $n$ *is a threshold number of* $\mathcal{X}$.

**Open Problem 4.** *Is there a known threshold number of* $\mathcal{P}_{n^2+1}$ *?*

Open Problem 4 asks about the existence of a year $t \geqslant 2023$ in which the implication $\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq (-\infty, n]$ will hold for some known integer $n$. We recall that $\mathcal{T}$ denote the set of twin primes.

**Open Problem 5.** *Is there a known threshold number of* $\mathcal{T}$ *?*

Open Problem 5 asks about the existence of a year $t \geqslant 2023$ in which the implication $\text{card}(\mathcal{T}) < \omega \Rightarrow \mathcal{T} \subseteq (-\infty, n]$ will hold for some known integer $n$.

# Competing interests

The author has declared that no competing interests exist.

# References


[1] Tyszka A. Statements and open problems on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that contain informal notions and refer to the current knowledge on $\mathcal{X}$. Creat. Math. Inform. 2023;32(2):247-253.
Available: http://semnul.com/creative-mathematics/wp-content/uploads/2023/07/creative_2023_32_2-_247_253.pdf

[2] Webb BLJ. Science, truth, and meaning: From wonder to understanding. World Scientific, Singapore; 2021.
Available: http://doi.org/10.1142/12152

[3] Case J, Ralston M. Beyond Rogers' non-constructively computable function. in: The nature of computation. Lecture Notes in Comput. Sci. Springer, Heidelberg. 2013;7921:45-54.
Available: http://link.springer.com/chapter/10.1007/978-3-642-39053-1_6

[4] How can it be decidable whether $\pi$ has some sequence of digits?.
Available: http://cs.stackexchange.com/questions/367/how-can-it-be-decidable-whether-pi-has-some-sequence-of-digits

[5] Rogers, Jr. H. Theory of recursive functions and effective computability. 2nd ed., MIT Press, Cambridge, MA; 1987.

[6] Sloane NJA. The on-line encyclopedia of integer sequences. A002496, Primes of the form $n^2+1$.
Available: http://oeis.org/A002496

[7] Sloane NJA. The on-line encyclopedia of integer sequences. A083844, Number of primes of the form $x^2+1<10^n$.
Available: http://oeis.org/A083844

[8] Wolfram Math World. Landau's Problems.
Available: http://mathworld.wolfram.com/LandausProblems.html

[9] Caldwell CK, Gallot Y. On the primality of $n! \pm 1$ and $2 \times 3 \times 5 \times \cdots \times p \pm 1$. Math. Comp. 2002;71(237):441-448.
Available: http://doi.org/10.1090/S0025-5718-01-01315-1

[10] Is $n!+1$ often a prime?.
Available: http://math.stackexchange.com/questions/853085/is-n-1-often-a-prime

[11] Factorial prime.
Available: http://en.wikipedia.org/wiki/Factorial_prime

[12] Nagell T. Einige Gleichungen von der Form $ay^2+by+c=dx^3$. Norske Vid. Akad. Skrifter, Oslo I. 1930;7.

[13] Sierpiński W. Elementary theory of numbers. 2nd ed. (ed. A. Schinzel). PWN - Polish Scientific Publishers and North-Holland, Warsaw-Amsterdam; 1987.

[14] Weisstein EW. CRC concise encyclopedia of mathematics. 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2002.

[15] Křížek M, Luca F, Somer L. 17 lectures on Fermat numbers: from number theory to geometry. Springer, New York; 2001.
Available: http://doi.org/10.1007/978-0-387-21850-2

[16] Ribenboim P. The little book of bigger primes. 2nd ed., Springer, New York; 2004.

[17] De Koninck JM, Luca F. Analytic number theory: Exploring the anatomy of integers. American Mathematical Society, Providence, RI; 2012.

[18] Lloyd S. Ultimate physical limits to computation. Nature. 2000;406:1047-1054.
Available: http://doi.org/10.1038/35023282



***Peer-review history:***
*The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)*
*http://www.sdiarticle5.com/review-history/110604*