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Bi-Metric Dimension of Graphs

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Abstract

For a connected graph G, a subset $S = \{s_1, s_2, \ldots, s_k\}$ of vertices of G and each vertex x of G we associate a pair of k-dimensional vectors (u, v), where $u = (d(x, s_1), d(x, s_2), \ldots, d(x, s_k))$ and $v = (\delta(x, s_1), \delta(x, s_2), \ldots, \delta(x, s_k))$, where $d(x, s_i)$ and $\delta(x, s_i)$ respectively denote the lengths of a shortest and longest paths between x and s_i. The subset S is said to bi-resolve G if no two distinct vertices receive the same pair. The minimum cardinality of a bi-resolving set is called bi-metric dimension of G. In this paper we show bi-metric dimension is lesser than or equal to the metric dimension and determine bi- metric dimensions of some standard graphs.

Keywords: Metric Dimension, Landmarks, Bi-Metric dimension
2010 Mathematics Subject Classification: 05C12; 05C20

1 Introduction

All the graphs considered in this paper are simple, connected and undirected. For any two vertices x and y, $d(x, y)$ and $\delta(x, y)$ respectively denote the length of the shortest and longest path between x and y and are called $distance$ and $detour\ distance$ between x and y. A subset S of the vertex set V of a connected graph G is said to be *r*esolving set of G if for every pair of vertices $u, v \in V - S$ there exists a vertex $w \in S$ such that $d(u, w) \neq d(v, w)$. The metric dimension of a graph G, denoted by $\beta(G)$, is the minimum cardinality of a resolving sets S of G. Metric dimension is defined independently by F. Harary et al. [1] and P.J. Slater [2], [3]. The terms not defined here may be found in [4], [5]. For the similar work on metric dimension we refer [6], [7], [8], [9], [10], [11].

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2 Bi-Metric Dimension

Let $G(V, E)$ be a simple connected graph. For each vertex $x \in V$, we associate a pair of vectors (u, v), denoted by S_x, with respect to a subset $S = \{s_1, s_2, \ldots, s_k\}$ of vertices of G where $u = (d(x, s_1), d(x, s_2), \ldots, d(x, s_k))$ and $v = (\delta(x, s_1), \delta(x, s_2), \ldots, \delta(x, s_k))$. The subset S is then said to bi-resolve G if $S_x \neq S_y$, whenever $x \neq y$. The minimum cardinality of a bi-resolving set S is termed as bi-metric dimension of G and is denoted by $\beta_b(G)$. The vertices in a minimal bi-resolving set S are called landmarks and the set S constitute a bi-metric basis for G. The bi-metric basis and dimension defined here also serve the same purpose of metric dimension introduced for the sake of unique addressing and locating property. In this paper we show that for certain families of graphs the bi-metric dimension is lesser than as that of metric dimension.

For example consider the graph G as shown in Figure 1. The set $S = \{v_1, v_2\}$ is a metric basis of G and the vectors assigned for each vertex with respect to S and the shortest distance d is also shown in the figure.

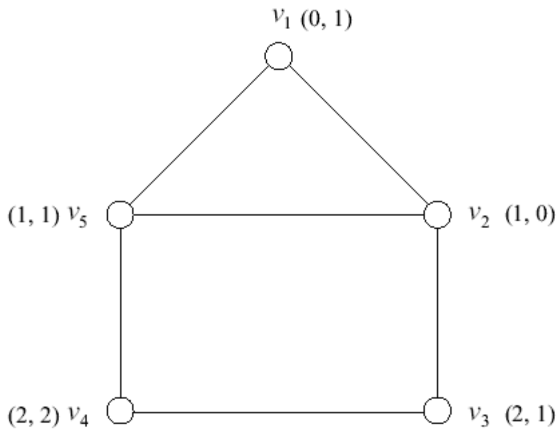

Figure 1: A graph G with metric dimension 2.

For the graph G shown in Figure 2, $S = \{v_1, v_2\}$ is a bi-metric basis and $\beta_b(G) = 2$. The pair of vector assigned for each vertex are also shown in the figure.

We recall the following results for immediate reference.

Theorem 2.1. *[12] For a graph G, $\beta(G) = 1$ if and only if G is a path on n vertices.*

Theorem 2.2. *[1] For any positive integer n, $\beta(G) = n - 1$ if and only if $G \cong K_n$.*

Theorem 2.3. *[13] A graph G with $\beta(G) = k$ cannot have $K_{3,3}$ if $k = 2$ and $K_{5\times 2^{k-3}, 5\times 2^{k-3}+1}$ if $k \geq 3$ or $K_{2^k+1} - (2^{k-1} - 1)e$ as a subgraph for any $k \geq 2$.*

Remark 2.1. In particular if the metric dimension of a graph is 2, then the above theorem tells that G should not contain a subgraph isomorphic to $K_5 - e$.

Theorem 2.4. *[6] If G is a connected graph of order n, then $\beta(G) \leq n - diam(G)$.*

We begin with the lemma whose proof is a direct consequence of the definition of bi-metric dimension and Theorem 2.4.

Lemma 2.5. *For any non-trivial connected graph G of order n, $1 \leq \beta_b(G) \leq \beta(G) \leq n - diam(G)$.*

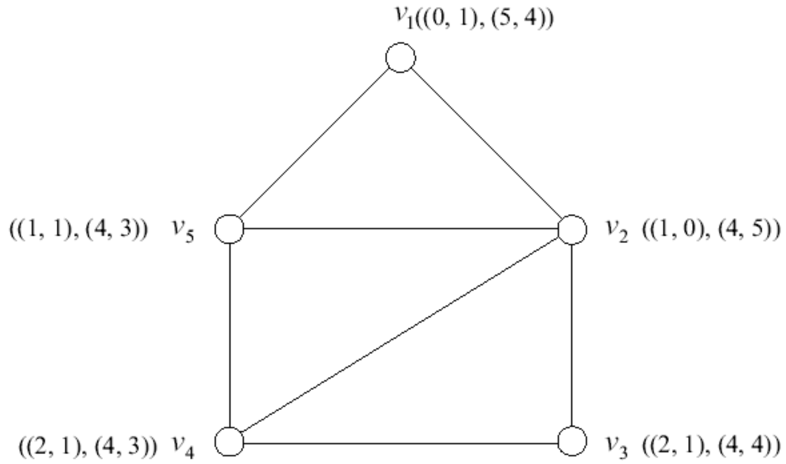

Figure 2: A graph G with bi-metric dimension 2.

By Lemma 2.5 it follows that, if G is path then $\beta_b(G) = 1$. Now consider a connected graph $G = (V, E)$, with $\beta_b(G) = 1$. Let $S = \{w\}$ be a bi-resolving set of G. If $deg(w) \geq 2$, then we can find at least two vertices u, v both are adjacent to w in G such that $\delta(w, u) = \delta(w, v)$. (If w is not in any cycle of G, then for any two vertices u, v both are adjacent to w we find $\delta(w, u) = \delta(w, v) = 1$ and if w is vertex in a cycle of G then we choose u, v both are adjacent vertices of w and which are in a largest cycle containing w). Which is a contradiction to the fact that S is a bi-resolving set. Thus $deg(w) = 1$.

Claim: There is no vertex of degree greater than two in G.

Assume that there are vertices of degree greater than two. Let w' be the nearest vertex to w such that $deg(w') \geq 3$. Then there exist two vertices u, v adjacent to w' such that $d(w, u) = d(w, v)$. (If w is not in any cycle of G, then for any two vertices u, v both are adjacent to w we find $\delta(w, u) = \delta(w, v) = d(w, w') + 1$ and if w is vertex in a cycle of G then we choose u, v both adjacent vertices of w and which are in a largest cycle containing w). Further, since we have $deg(w) = 1$ and w' is the nearest vertex of degree greater than two, we must have $\delta(w, u) = \delta(w, v)$ which is a contradiction to the fact that S is bi-resolving set of G. Hence the claim.

Thus G is a connected graph in which w is vertex of degree one and all other vertices are of degree less than three, hence G must be a path. Thus we have proved the following theorem.

Theorem 2.6. *For a non-trivial graph G, $\beta_b(G) = 1$ if and only if G is a path.*

In a non-trivial complete graph, $d(x, y) = 1$ and $\delta(x, y) = n - 1$, for each pair of vertices x and y, hence it follows that $\beta_g(K_n) \geq n - 1$ whenever $n \geq 2$. Thus, by Lemma 2.5, we get $\beta_b(K_n) = n - 1$ for all $n \geq 2$. For a nontrivial connected graph G, if $\beta_b(G) = n - 1$, then as $n - 1 = \beta_b(G) \leq \beta(G) \leq n - 1$, it follows that $\beta(G) = n - 1$, which is possible if and only if $G \equiv K_n$. Hence we conclude;

Theorem 2.7. *For a non-trivial connected graph G on n vertices, $\beta_b(G) = n - 1$ if and only if G is a complete graph K_n.*

For each pair of vertices x and y in the cycle C_n, $\delta(x, y) = n - d(x, y)$, wheel W_n, $\delta(x, y) = n - 1$ and for tree T, $\delta(x, y) = d(x, y)$. Therefore for the graphs C_n, W_n and a tree T we observe that bi-metric dimension is equal to metric dimension.

Now we see the cases where the Bi-metric dimension is strictly less than the metric dimension. The graphs in Figure 3 are some examples of graphs for which bi-metric dimension is strictly less than metric dimension. In fact, for the graphs G_1 and G_2, the set $S = \{v_1, v_2, v_3, v_5\}$ is a minimum resolving set and hence their metric dimensions is 4. But, the set $S = \{v_2, v_3, v_6\}$ is a minimum bi-resolving set for both the graphs. Thus, bi-resolving dimension is 3.

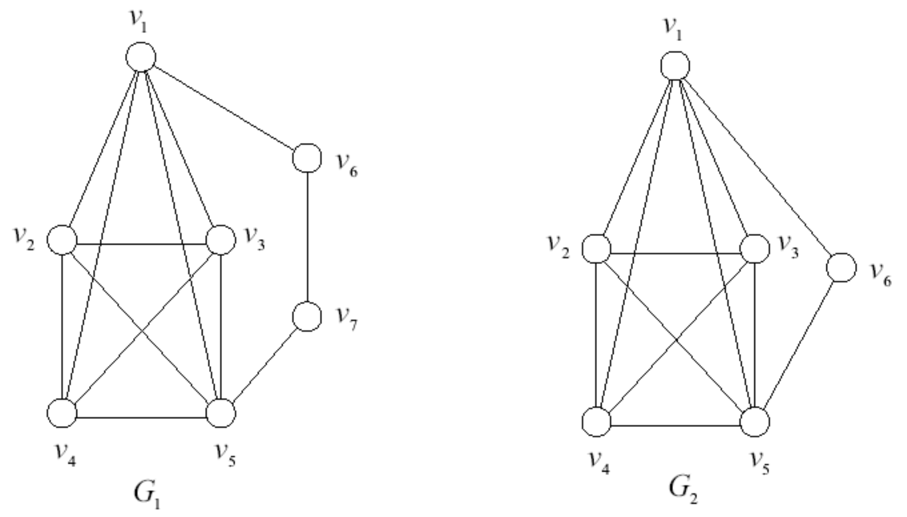

Figure 3: The graphs having Bi-metric dimension less than metric dimension

3 Complexity in determining Bi-resolving Sets

We now see the complexity in computation of the bi-metric dimension of a graph G. Consider the graphs G_1 and G_2 shown in the Figure 4. The Bi-metric dimension of G_1 is 2 with a minimum bi-resolving set $S = \{v_1, v_2\}$ and that of the graph G_2, obtained from G_1 by adding an edge v_1v_3, is also 2 with the same bi-resolving set. Thus, addition of an edge does not alter the bi-metric dimension in this case. However it is not true in general, for example, the bi-metric dimension of $K_4 - e$ is 2 (see graph G_2 of figure 3) and K_4 is 3 (by Theorem 2.2).

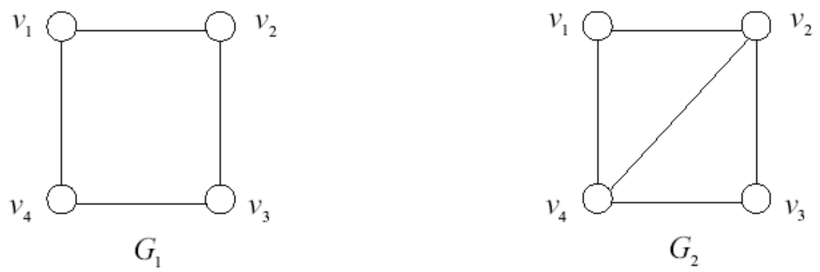

Figure 4: Graph G_1 and G_2 with bi-metric dimension 2.

Further, by adding an edge to the graph, its bi-metric dimension may decrease also. For example we see that for the graph G_3 shown in the Figure 5, the bi-metric dimension of is 3 and that of graph

G_4 obtained from G_3 by adding the edge v_5v_1 is 2. A similar set of graphs can also be obtained for

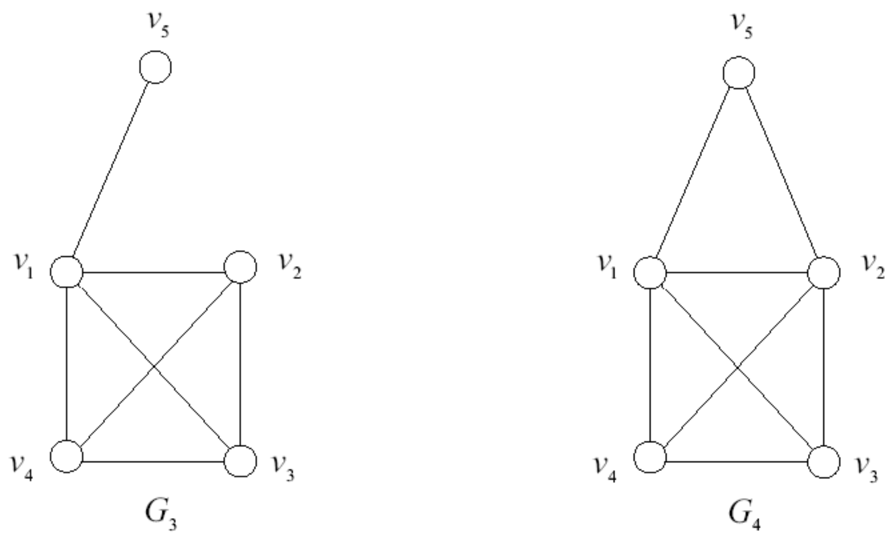

Figure 5: Graphs G_3 and G_4 with $\beta_b(G_3) = 3$ and $\beta_b(G_4) = 2$.

the addition of vertices. Thus, determination of a bi-metric dimension for a given graph cannot be derived from any graph of higher or lower dimensions. At the first glance it appears that, an edge common to two cycles in a graph is a necessary condition for bi-metric dimension to be lesser than metric dimension. But we can find examples of mono cyclic graphs with bi-metric dimension lesser than metric dimension. For the graph G in Figure 6, $S = \{v_2, v_3\}$ is a bi-resolving set, therefore $\beta_b(G) = 2$, where as $\beta(G) = 3$ with minimum a resolving set $S = \{v_3, v_5, v_6\}$.

Comparing with metric dimension, bi-metric dimension varies in characterization. Unlike graphs with metric dimension 2, graphs with bi-metric dimension 2 can contain K_5 or $K_{3,3}$ as subgraphs. For the graph G_5 shown in Figure 7, which has K_5 as its subgraph, $S = \{v_2, v_5\}$ is a bi-resolving set, and $\beta_b(G_5) = 2$. For the graph G_6 shown in Figure 7, which has $K_{3,3}$ as its subgraph, $S = \{v_2, v_5\}$ is a bi-resolving set, and $\beta_b(G_6) = 2$.

4 Graphs with $\beta_b(G) \leq \frac{\beta(G)}{2}$

The purpose of finding a minimal resolving set and metric dimension of a graph can be met much effectively through a minimal bi-resolving set and bi-metric dimension. Further a graph with bi-metric dimension less than or equal to half of metric dimension is of greater importance, as the length of the code of each vertex is less than that of metric basis. This will lead to reduction of cost of establishing a network with navigation agents to nearly half of the original cost. We can establish a network with desired minimum number of vertices having bi-metric dimension less than or equal to half of metric dimension. We can reduce the bi-metric dimension of certain graphs to half the metric dimension by superimposing it into another graph of slightly higher order.

For any integer $n \geq 4$, let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_n. Define a graph G_n^* by

$$V(G_n^*) = V(K_n) \bigcup \left\{w_1, w_2, \ldots, w_{\lfloor \frac{n}{2} \rfloor}\right\}$$

$$E(G_n^*) = E(K_n) \bigcup \left\{w_i v_{2i-1}, w_i v_{2i} : i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$$

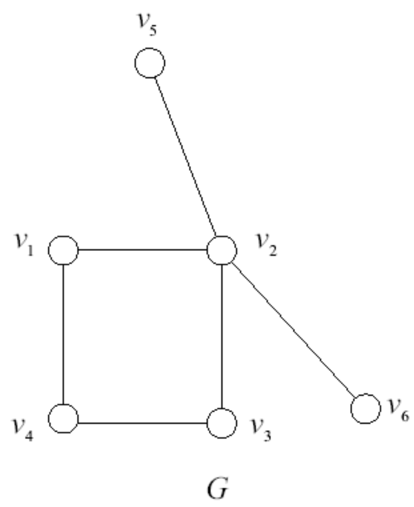

Figure 6: The Graph G with single cycle, having $\beta_b(G) = 2$ and $\beta(G) = 3$.

The graphs G_4^* and G_5^* are shown in the Figure 8.

Observation 4.1. *In the graph G_n^* as for $i \neq j$, $d(v_i, v_j) = 1$ and $d(w_i, w_j) = 3$.*

$$d(v_i, w_j) = \begin{cases} 1 & for\ j = i\ or\ j = i + 1 \\ 2 & otherwise \end{cases}$$

Lemma 4.1. *Let S be a resolving set of G_n^* and $\{v_1, v_2, \ldots, v_n\}$ be the vertices of G_n^* corresponding to vertices of K_n. Then one of v_{2i-1} or v_{2i} must belong to S for each i, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. Consider pair of vertices v_{2i-1}, v_{2i} where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, we find $d(v_{2i-1}, w_i) = d(v_{2i}, w_i) = 1$, $d(v_{2i-1}, w_j) = d(v_{2i}, w_j) = 3$, for all $j \neq i$, $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ and $d(v_{2i-1}, v_k) = d(v_{2i}, v_k) = 1$. Thus for any $w \in V - \{v_{2i-1}, v_{2i}\}$, $d(v_{2i-1}, w) = d(v_{2i}, w)$. Hence no vertex in the set $V - \{v_{2i-1}, v_{2i}\}$ can resolve v_{2i-1} and v_{2i}. Therefore one of v_{2i-1} or v_{2i} must belong to S. ☐

Theorem 4.2. *For any integer $n \geq 4$, $\beta(G_n^*) = n - 1$.*

Proof. Let S be a metric basis for G_n^*. Then by Lemma 4.1, one of v_{2i-1} or v_{2i} is in S for each i, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Since v_{2i-1} and v_{2i} are vertices of K_n which are the only vertices adjacent to w_i in G_n^*, without loss of generality, we consider $v_{2i} \in S$ for each i, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Consider the pair of vertices v_{2j-1}, v_{2j+1} for some j, $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$. Then $d(v_{2j-1}, v_k) = d(v_{2j+1}, v_k) = 1$ for all k, $1 \leq k \leq n$ and $d(v_{2j-1}, w_l) = d(v_{2j+1}, w_l) = 2$ for all $l \neq j$, $1 \leq l \leq \lfloor \frac{n}{2} \rfloor$. Hence for each j, $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ one of v_{2j-1}, v_{2j+1}, w_j or w_{j+1} must belongs to S. Thus for each i, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $v_2 i \in S$ and one of v_{2i-1}, v_{2i+1}, w_i or w_{i+1} must belong to S. Thus $|S| \geq n - 1$

To prove the reverse inequality, consider the set $S = \{v_1, v_2, \ldots, v_{n-1}\}$. For this set S, $V(G_n^*) - S = \{v_n, w_1, w_2, \ldots, w_n\}$ and for any pair of vertices $w_i, w_j \in V(G_n^*) - S$, we find $v_{2i-1} \in S$ such that $d(w_i, v_{2i-1}) = 1$ and $d(w_j, v_{2i-1}) = 2$. For pair of vertices $v_n, w_i \in V(G_n^*) - S$ we find one of $d(v_1, w_i)$ or $d(v_3, w_i)$ is 2, but $d(v_n, v_i) = 1$ for $i \neq n$. Therefore S is a resolving set with $|S| = n - 1$. Hence $\beta(G_n^*) = n - 1$. ☐

Lemma 4.3. *Let S be a bi-resolving set of G_n^* and $\{v_1, v_2, \ldots, v_n\}$ be the vertices of G_n^* corresponding to vertices of K_n. Then one of v_{2i-1} or v_{2i} must belong to S for each i, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.*

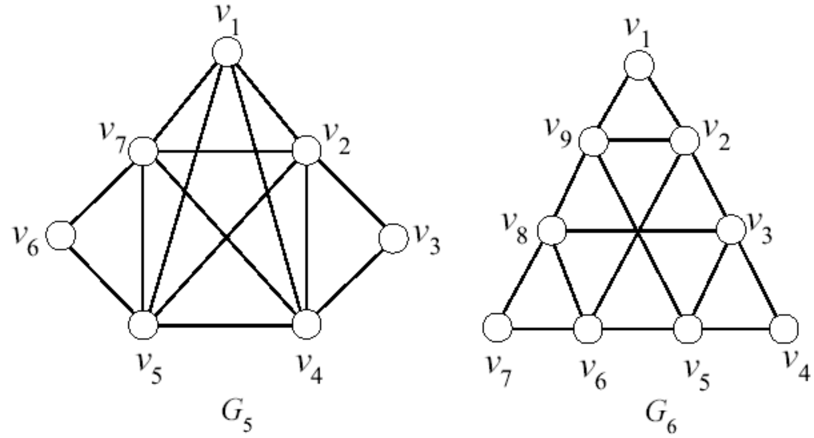

Figure 7: Graphs G_5 and G_6, with $\beta_b(G_5) = \beta_b(G_6) = 2$.

Proof. For any vertex $w_j \in V(G_n^*)$ we observe;

$$d(w_j, v_{2i-1}) = d(w_j, v_{2i}) = \begin{cases} 1 & \text{if } i = j \\ 2 & \text{otherwise} \end{cases}$$

and

$$\delta(w_j, v_{2i-1}) = \delta(w_j, v_{2i}) = \begin{cases} n + \lfloor \frac{n}{2} \rfloor - 2 & \text{if } j = i \\ n + \lfloor \frac{n}{2} \rfloor - 1 & \text{otherwise} \end{cases}$$

Hence none of the vertices w_j resolves pair of vertices v_{2i-1}, v_{2i}. For any $j \notin \{2i-1, 2i\}$, we find $d(v_j, v_{2i-1}) = d(v_j, v_{2i}) = 1$ and $\delta(v_j, v_{2i-1}) = \delta(v_j, v_{2i}) = n + \lfloor \frac{n}{2} \rfloor - 1$. Thus none of the vertices v_j, $j \notin \{2i-1, 2i\}$ resolves pair of vertices v_{2i-1}, v_{2i}. Therefore for each i, $1 \le i \le \lfloor \frac{n}{2} \rfloor$ one of the vertices v_{2i-1} or v_{2i} must belong to S. ☐

Theorem 4.4. *For any integer $n \ge 4$, $\beta_b(G_n^*) = \lfloor \frac{n}{2} \rfloor$.*

Proof. Let S be a resolving set of G_n^*. By the Lemma 4.3, for each i, $1 \le i \le \lfloor \frac{n}{2} \rfloor$ one of the vertices v_{2i-1} or v_{2i} must belongs to S. Hence $|S| \ge \lfloor \frac{n}{2} \rfloor$ and $\beta_b(G_n^*) \ge \lfloor \frac{n}{2} \rfloor$. Consider $S = \{v_1, v_3, v_5, \ldots, v_{2\lfloor \frac{n}{2} \rfloor - 1}\}$. Then $|S| = \lfloor \frac{n}{2} \rfloor$. It is enough to prove S is a bi-resolving set of G_n^*.

For each $v_i \in S$ $\;d(v_i, v_j) = 1$ for $i \ne j$,

$$d(v_i, w_j) = \begin{cases} 1 & \text{if } i = 2j-1 \\ 2 & \text{otherwise} \end{cases},$$

$$\delta(v_i, v_j) = \begin{cases} n + \lfloor \frac{n}{2} \rfloor & \text{for } j = i \\ n + \lfloor \frac{n}{2} \rfloor - 2 & \text{for } j = i+1 \\ n + \lfloor \frac{n}{2} \rfloor - 1 & \text{otherwise} \end{cases}$$

and $\delta(v_i, w_j) = n + \lfloor \frac{n}{2} \rfloor - 1$ for all j. Thus for $i = 2k-1$, v_i resolves v_{i+1} and w_k with the remaining vertices of G_n^*. Hence S is a bi-resolving set and $\beta_b(G_n^*) = \lfloor \frac{n}{2} \rfloor$. ☐

From Theorem 4.2 and 4.4 we conclude that, for any integer $n \ge 5$, $\beta_b(G_n^*) = \left\lceil \frac{\beta(G_n^*)}{2} \right\rceil$.

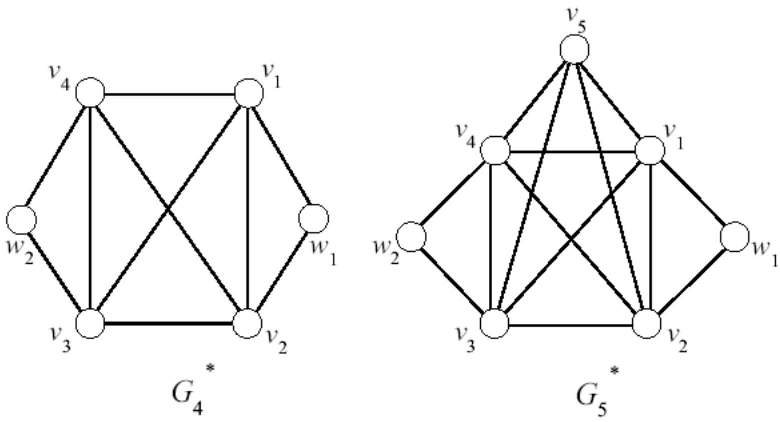

Figure 8: The graphs G_4^* and G_5^*, obtained from K_4 and K_5 respectively

5 Metric dimension of $K_1 \odot P_n$

Let G and H be two graphs of order n_1 and n_2, respectively. The corona product $G \odot H$ is the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G. The graph $C_3 \odot P_2$ is shown in Figure 9.

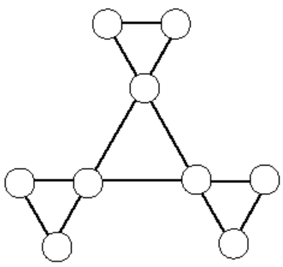

Figure 9: The Graph $C_3 \odot P_2$.

The graph $K_1 \odot P_n$ is of order $n+1$, in which the vertex of K_1 is adjacent to every vertex of path P_n. In this section we determine metric dimension of $K_1 \odot P_n$ for all positive integers n. Through out this session, we consider $G = K_1 \odot P_n$ with $\{v_1, v_2, v_3, \ldots, v_n, v\}$ being the vertex set of G where $v_1, v_2, v_3, \ldots, v_n$ correspond to vertices of P_n such that $d(v_i, v_{i+1}) = 1$ for each i, $1 \leq i \leq n-1$ and v corresponds to the vertex of K_1.

Lemma 5.1. *For $n \geq 4$, if S is metric basis of $K_1 \odot P_n$, then $\{v_1, v_2, v_3\} \cap S \neq \phi$.*

Proof. Let S be metric basis of $K_1 \odot P_n$. We note that $d(v_1, v_i) = d(v_2, v_i) = 2$ for all $i > 3$ and $d(v_1, v) = d(v_2, v) = 1$. Thus for $i > 3$, v_i as well as v does not resolve pair of vertices v_1, v_2. Therefore one of v_1, v_2, v_3 must belong to S and hence $\{v_1, v_2, v_3\} \cap S \neq \phi$. □

Lemma 5.2. *For $n \geq 7$, if S is metric basis of $K_1 \odot P_n$, then $\{v_1, v_2, v_{n-1}, v_n\} \cap S \neq \phi$.*

Proof. Let S be metric basis of $K_1 \odot P_n$. For any i, $3 \leq i \leq n-2$, $d(v_1, v_i) = d(v_n, v_i) = 2$. Thus for $3 \leq i \leq n-2$, v_i cannot resolve pair of vertices v_1, v_n. Therefore one of v_1, v_2, v_{n-1}, v_n must belong to S and hence $\{v_1, v_2, v_{n-1}, v_n\} \cap S \neq \phi$. □

Remark 5.1. In P_n we can rename vertices in reverse order if required and above lemma can be stated as, "If S is a resolving set of $K_1 \odot P_n$, then $\{v_1, v_2\} \cap S \neq \phi$ or $\{v_{n-1}, v_n\} \cap S \neq \phi$".

Remark 5.2. For $n \geq 6$, by the Lemma 5.1 and Lemma 5.2, and in view of the remark 5.1 we conclude that "If S is a resolving set of $K_1 \odot P_n$, then among $v_1, v_2, v_3, v_{n-2}, v_{n-1}, v_n$ at least two vertices must belong to S".

Lemma 5.3. *Let S be a resolving set of $K_1 \odot P_n$ such that $\{v_i, v_{i+2}\} \cap S = \phi$ for some i, $1 < i < n$, then $\{v_{i-1}, v_{i+3}\} \cap S \neq \phi$*

Proof. Let $\{v_i, v_{i+2}\} \cap S = \phi$ for some i, $1 < i < n$. Suppose $\{v_{i-1}, v_{i+3}\} \cap S = \phi$, then $d(v_k, v_{i-1}) = d(v_k, v_{i+1}) = 2$ for each vertex $v_k \in S$. Which is a contradiction to that S is a resolving set. Hence $\{v_{i-1}, v_{i+3}\} \cap S \neq \phi$. □

Lemma 5.4. *If S be a resolving set of $K_1 \odot P_n$ then for each i, $1 < i < n-1$, $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\} \cap S \neq \phi$*

Proof. Let S be a resolving set of $K_1 \odot P_n$. Suppose $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\} \cap S = \phi$ for some i, $1 < i < n-1$, then $d(v_{i+1}, v_k) = d(v_{i+2}, v_k) = 2$ for each vertex $v_k \in S$. A contradiction to the fact that S is a resolving set and hence the lemma. □

Remark 5.3. In view of the Lemma 5.3 and Lemma 5.4 we can conclude that "If S is a resolving set of $K_1 \odot P_n$, then $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\} \cap S \neq \phi$ for each i, $1 < i < n$".

Lemma 5.5. *If S is a metric basis of $K_1 \odot P_n$ with $|S| \geq 3$, then $v \notin S$.*

Proof. Let S be a resolving set of $K_1 \odot P_n$ and $v \in S$. By the Theorem 2.6, $|S| \geq 2$. Since $d(v, v_i) = d(v, v_j) = 1$ for each i, j, $1 \leq i, j \leq n$, v cannot resolve any pair of vertices $v_i, v_j \in V - S$. Thus v resolve only the pair of vertices v, v_i, for all $v_i \in V - S$. If $v_i \in V - S$ and $d(v, w) = d(v_i, w)$ for all $w \in S - \{v\}$, then every vertex in S must be adjacent to v_i because $d(v, w) = 1$. Which is not possible when $|S| \geq 3$, since v_i can be adjacent to at most two vertices of P_n. □

Theorem 5.6. *For any integer $n \geq 1$ we have,*

$$\beta(K_1 \odot P_n) = \begin{cases} 1, & \textit{if} \quad n = 1 \\ 2, & \textit{if} \quad 2 \leq n \leq 5 \\ 3, & \textit{if} \quad n = 6 \\ \left\lfloor \frac{2n+2}{5} \right\rfloor, & \textit{if} \quad n \geq 7 \end{cases}$$

Proof. Let $G = K_1 \odot P_n$ with $\{v_1, v_2, v_3, \ldots, v_n, v\}$ being the vertex set of G where $v_1, v_2, v_3, \ldots, v_n$ correspond to vertices of P_n and v corresponds to the vertex of K_1.

If $n = 1$ or $n = 2$, then G is a complete graph and the result is obvious.

If $n \in \{3, 4, 5\}$, then C_3 is a subgraph of G and hence $\beta(G) \geq 2$. On the other hand, in these cases $S = \{v_2, v_3\}$ is a resolving set of G and hence $\beta(G) = 2$.

If $n = 6$, then clearly $\beta(G) \geq 2$. Let S be any subset of $V(G)$ with $|S| = 2$. Then we claim that "S is not a resolving set of G". Let $S = \{v, v_k\}$ for some k, $1 \leq k \leq 6$. Then for every vertex $v_i \in V - S$, $d(v, v_i) = 1$ and $d(v_k, v_{k+2}) = d(v_k, v_{k+3}) = 2$ if $k \leq 3$ and $d(v_k, v_{k-2}) = d(v_k, v_{k-3}) = 2$ when $k > 3$. Hence S is not a resolving set of G.

In case if $v \notin S$, say $S = \{v_i, v_j\}$, for some $i < j$. For $j - i = 1$ or 5 then there exist two vertices v_l *and* v_m such that $d(v_l, w) = d(v_m, w) = 2$ for every $w \in S$ and hence S is not a resolving set. If

$j - i = 2$ then vertices v_{i+1} and v are adjacent to every vertex in S and hence S is no a resolving set of G. In case if $j - i \geq 3$ or 4, then two vertices v_{i-1} and v_{i+1} (or v_{j-1} and v_{j+1}) are equidistant from every vertex in S and hence S is not a resolving set of G. Hence the claim. Therefore if S is a resolving set of G, then $|S| \geq 3$ and hence $\beta(G) \geq 3$. Now we choose $S = \{v_1, v_4, v_6\}$, then S is a resolving set of G and hence $\beta(G) = 3$ in this case.

Now we consider the case $n = 7$. By the Remark 5.2, among v_1, v_2, v_3, v_5, v_6 and v_7 at least two vertices must belong to S. Also by the Lemma 5.1 and Lemma 5.2, and the symmetry of the graph it is enough to discus the following cases. If $v_1, v_5 \in S$ or $v_2, v_5 \in S$, then to resolve pair of vertices v_4, v_6, we require one more vertex in S. If $v_1, v_6 \in S$, or $v_1, v_7 \in S$, then to resolve pair of vertices v_3, v_4, we require one more vertex in S. Thus if $|S|$ is resolving set then $|S| \geq 3 = \lfloor \frac{2n+2}{5} \rfloor$. The set $S = \{v_1, v_4, v_6\}$ is a resolving set for $K_1 \odot P_7$. Hence result holds in this case.

Finally when $n > 7$, if S is any resolving set of $K_1 \odot P_n$, then first we prove that $|S| \geq \lfloor \frac{2n+2}{5} \rfloor$ and later we construct a resolving set S of desired cardinality.

Case 1: $n \equiv 0\ (mod\ 5)$.

In view of the Remark 5.3, we require at least $\frac{2n}{5}$ vertices in S to resolve every pair of vertices in V. Thus $|S| \geq \frac{2n}{5}$. If $n \equiv 0\ (mod\ 5)$, then $\frac{2n}{5} = \lfloor \frac{2n+2}{5} \rfloor$. Hence $|S| \geq \lfloor \frac{2n+2}{5} \rfloor$.

Case 2: $n \equiv 1\ (mod\ 5)$.

If $n \equiv 1\ (mod\ 5)$, then $n - 1 \equiv 0\ (mod\ 5)$ and $\frac{2(n-1)}{5} = \lfloor \frac{2n+2}{5} \rfloor$. Since $n - 1 \equiv 0\ (mod\ 5)$, in view of the Remark 5.3, we require at least $\frac{2(n-1)}{5}$ vertices in S to resolve every pair of vertices in V. Hence $|S| \geq \lfloor \frac{2n+2}{5} \rfloor$.

Case 3: $n \equiv 2\ (mod\ 5)$.

If $n \equiv 2\ (mod\ 5)$, then by the Remark 5.3, and by the Lemma 5.2 we require at least $\frac{2(n-2)}{5}$ vertices in S to resolve every pair of vertices in $\{v_1, v_2, v_3, \ldots, v_{n-3}\} \subset V$. Then by the Lemma 5.1, at least one of the vertices in v_{n-2}, v_{n-1}, v_n must be in S. Hence $|S| \geq \left\lfloor \frac{2(n-2)}{5} \right\rfloor + 1 = \lfloor \frac{2n+2}{5} \rfloor$.

Case 4: $n \equiv 3\ (mod\ 5)$.

If $n \equiv 3\ (mod\ 5)$, then $n - 3 \equiv 0\ (mod\ 5)$ and by the Remark 5.3, and by the Lemma 5.2 we require at least $\frac{2(n-3)}{5}$ vertices in S to resolve every pair of vertices in $\{v_1, v_2, v_3, \ldots, v_{n-3}\} \subset V$. Then by the Lemma 5.1, at least one of the vertices in v_{n-2}, v_{n-1}, v_n must be in S. Hence $|S| \geq \left\lfloor \frac{2(n-3)}{5} \right\rfloor + 1 = \frac{2n-1}{5} = \lfloor \frac{2n+2}{5} \rfloor$.

Case 5: $n \equiv 4\ (mod\ 5)$.

In view of the Remarks 5.1, 5.2, and by the Lemma 5.4 we find that, if S is a resolving set, then between two consecutive pairs of vertices belonging to S, there will be alternately 1 and 2 vertices of $V - S$. Therefore a Bi-resolving set S can be one of the following.

1. $S = \{v_1, v_3, v_6, v_8, v_{11}, \ldots, v_{n-3}, v_n\}$.
2. $S = \{v_1, v_4, v_6, v_9, v_{11}, \ldots, v_{n-3}, v_n\}$.
3. $S = \{v_2, v_5, v_7, v_{10}, v_{12}, \ldots, v_{n-5}, v_{n-2}\}$.
4. $S = \{v_2, v_4, v_7, v_9, v_{12}, \ldots, v_{n-5}, v_{n-2}\}$
5. $S = \{v_3, v_5, v_8, v_{10}, v_{13}, \ldots, v_{n-4}, v_{n-1}, v_n\}$.

In any of the case $|S| = \frac{2n+2}{5}$.

Now consider the set $S = \{v_3, v_5, v_8, v_{10}, v_{13}, v_{15}, \ldots, v_k\} \subseteq V$, where $k = n$, if $n \equiv 0, 3, 4$ (mod 5) and $k = n - 2$, if $n \equiv 1, 2$ (mod 5).

Let $v_i, v_j \in V - S, 3 < i < j < n-3$. By the choice of S, either $v_{i-1} \in S$ or $v_{j+1} \in S$. If $v_{i-1} \in S$, then $d(v_{i-1}, v_i) = 1$ and $d(v_{i-1}, v_j) = 2$, hence v_{i-1} resolves pair of vertices v_i and v_j. If $v_{j+1} \in S$, then $d(v_{j+1}, v_i) = 2$ and $d(v_{j+1}, v_j) = 1$, hence v_{j+1} resolves pair of vertices v_i and v_j in this case. v_3 resolves pair of vertices v_1 and v_2. By the Remark 5.1, one of v_{n-i} or v_n must be in S. Which resolves the pair of vertices v_n and v_{n-1}. Hence S is a resolving set with desired cardinality. □

6 Bi-metric dimension of $K_1 \odot P_n$

In most of the applications, the network system constructed will be looked as corona product or a general corona product of two graphs. Here a main network consists of subnetworks and in each subnetwork we have to locate navigators. The problem of minimizing the number of navigators and thereby optimizing the cost reduction is seriously thought about. In this way the bi-metric dimension of a corona product or general corona product of paths will be of special interest. In this section we completely determine bi-metric dimension of graph $K_1 \odot P_n$ for all positive integer n.

Lemma 6.1. *Let $G = K_1 \odot P_n$, $\{v_1, v_2, \ldots, v_n\}$ be the vertices of P_n, such that v_i is adjacent to v_{i+1} for each $i = 1, 2, \ldots, n-1$ and v be the vertex corresponding to K_1, then*

$$\delta(v, v_i) = \begin{cases} n+1-i, & for \quad i \leq \frac{n}{2} \\ i, & for \quad i > \frac{n}{2} \end{cases}$$

and for any $i < j$,

$$\delta(v_i, v_j) = n + 1 - min\{i, j-i, n-j+1\}$$

Proof. The graph $G = K_1 \odot P_n$ is hamiltonian having a hamiltonian cycle $C_n : v - v_1 - v_2 - \cdots - v_n - v$. The longest path from vertex v to the vertex v_i is the part of this cycle. For $i \leq \frac{n}{2}$, the longest path is $v - v_n - v_{n-1} - \cdots - v_i$ and the length of the path is $n+1-i$. When $i > \frac{n}{2}$, the longest path from vertex v to v_i is $v - v_1 - v_2 - \cdots - v_i$ and the length of the path is i. To find $\delta(v_i, v_j)$, $i < j$, we divide the path, $v_1, v_2, \ldots, v_n$ into three parts P_1, a path from v_1 to v_i, P_2 a path from v_i to v_j and P_3 a path from v_j to v_n.

Case 1: Length of P_1 is least. In this case the longest path from v_i to v_j is $v_i - v_{i+1} - \cdots - v_{j-1} - v - v_n - v_{n-1} - \cdots v_j$ and length of the path is $n+1-i$ (as shown in Figure 10).

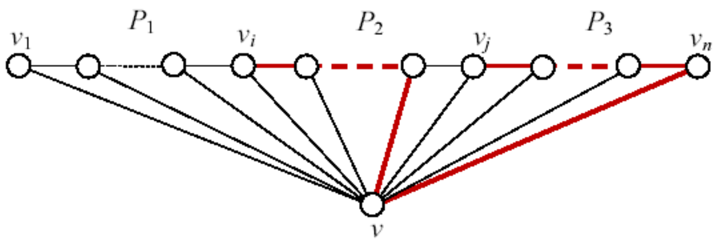

Figure 10: Longest path from vertex v_i to vertex v_j in $K_1 \odot P_n$.

Case 2: Length of P_2 is least. In this case the path $v_i - v_{i-1} - \cdots v_1 - v - v_{n-1} - \cdots - v_j$ is the longest path from v_i to v_j and the length of the path is $i + 1 + (n-j) = (n+1) - (j-1)$. (Figure 11)

Case 3: Length of P_3 is least. In this case the path $v_i - v_{i-1} - \cdots v_1 - v - v_{i+1} - v_{i+2} - \cdots - v_j$ is the longest path from v_i to v_j and the length of the path is $i + 1 + (j-i-1) = j = (n+1) - (n-j+1)$(Figure 12).

This completes the proof. ☐

Remark 6.1. Let $G = K_1 \odot P_n$, $\{v_1, v_2, \ldots, v_n\}$ be the vertices of P_n such that v_i is adjacent to v_{i+1} for each $i = 1, 2, \ldots, n-1$ and v be the vertex corresponding to K_1, then $\delta(v, v_i) = \delta(v, v_{n-i})$ (follows by the Lemma 6.1).

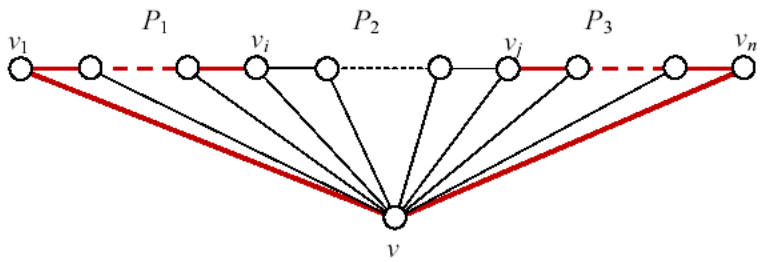

Figure 11: Longest path from vertex v_i to vertex v_j in $K_1 \odot P_n$.

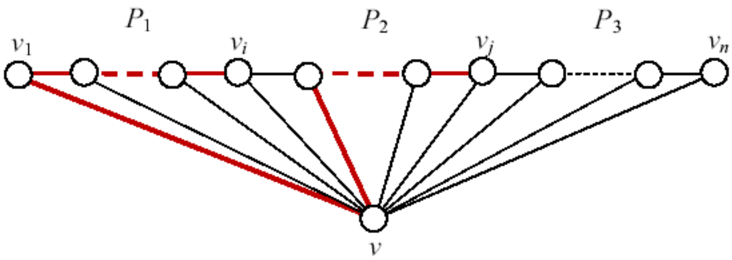

Figure 12: Longest path from vertex v_i to vertex v_j in $K_1 \odot P_n$.

Remark 6.2. For any fixed v_i,

$$\delta(v_i, v_j) = \begin{cases} n+1-j, & for \quad 1 \le j \le \lceil \frac{i}{2} \rceil \\ n+1+j-i, & for \quad \lceil \frac{i}{2} \rceil < j \le i \\ n+1-j+i, & for \quad i \le j \le i\lceil \frac{n-1}{2} \rceil \\ j, & for \quad \lceil \frac{n-1}{2} \rceil \le j \le n. \end{cases}$$

Theorem 6.2. *For any integer $n \ge 6$, we have*

$$\beta_b(K_1 \odot P_n) \le \left\lceil \frac{n}{6} \right\rceil + 1$$

Proof. Let $G = K_1 \odot P_n$, $\{v_1, v_2, \ldots, v_n\}$ be the vertices of P_n such that v_i is adjacent to v_{i+1} for each $i = 1, 2, \ldots, n-1$ and v be the vertex corresponding to K_1. Let $S = \{v, v_2, v_5, \ldots, v_k$: where $k = 3\lceil \frac{n}{6} \rceil - 1\}$. Then $|S| = \lceil \frac{n}{6} \rceil + 1$. We prove that S is bi-resolving set of G.

For any pair of vertices $v_i, v_j \in V(G) - S$, if $\delta(v, v_i) = \delta(v, v_j)$, then $i = n+1-j$, and one of i and j is less than $\lceil \frac{n}{2} \rceil$. Without loss of generality say $i < \lceil \frac{n}{2} \rceil$. By the choice of set S we observe that one of v_{i-1} and v_{i+1} must belong to S and $d(v_i, v_{i-1}) = d(v_i, v_{i+1}) = 1$, where as $d(v_j, v_{i-1}) = d(v_j, v_{i+1}) = 2$. Hence S is a bi-resolving set of G. □

Conjecture 6.3. *For any integer $n \ge 6$, we have*

$$\beta_b(K_1 \odot P_n) = \left\lceil \frac{n}{6} \right\rceil + 1$$

7 Bi-metric dimension of $P_m \odot P_n$

Consider the graph $P_m \odot P_n$, where $n \ge 6$. Let $\{v_1, v_2, v_3, \ldots, v_m\}$ be the vertices of P_m, such that v_i is adjacent to v_{i+1}, $1 \le i \le m$. For $i \le k \le m$, $\{w_{k_1}, w_{k_2}, w_{k_3}, \ldots, w_{k_n}\}$ be the vertices of k^{th} copy of P_n, such that w_{k_i} is adjacent to $w_{k_{i+1}}$ for each i, $1 \leq i \leq n$. The set $S = \{v_1\} \cup_{k=1}^{n} \{w_{k_2}, w_{k_5}, \ldots, w_{k_l} : l = 3\left\lceil \frac{n}{6} \right\rceil - 1\}$ is a bi-resolving set because, as in the proof of Theorem 6.2, $\{w_{k_2}, w_{k_5}, \ldots, w_{k_l} : l = 3\left\lceil \frac{n}{6} \right\rceil - 1\}$ along with v_1 bi-resolve the vertices of k^{th} copy of P_n and v_1 resolves vertices of P_m. Thus we have proved the following.

Corollary 7.1. *For any positive integer m and an integer $n \geq 6$, we have*

$$\beta_b(P_m \odot P_n) \leq m\left\lceil \frac{n}{6} \right\rceil + 1$$

8 Bi-Metric Dimension of Power of a Graph

The k^{th} power of a graph G, denoted by G^k, is defined on the vertices of G, with the property that two vertices in G are adjacent whenever $d_G(u, v) \leq k$. If d is diameter of G then we note that G^d is a complete graph and $G^1 = G$. In this section we determine bi-metric dimension of powers Paths and cycles.

Lemma 8.1. *For any positive integer n, the length of longest path between any two vertices v_i and v_j of P_n^2 is*

$$\delta(v_i, v_j) = \begin{cases} n, & \textit{if } i = j \\ max\{i-1, n-i+1\}, & \textit{if } i = j-1 \textit{ or } j+1 \\ n-1, & \textit{otherwise} \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n such that v_i is adjacent to v_{i+1} for $1 \leq i \leq n-1$. Consider $G = P_n^2$ and v_i, v_j be any two vertices of G with $i \leq j$. Since the graph P_n^2 is hamiltonian, $\delta(v_i, v_i) = n$. Let $i = j-1$. If i is odd then the path v_i, v_{i-2}, v_{i-4}, $\ldots, v_3$, v_1, v_2, v_4, $\ldots$, v_{i-1}, v_{i+1} is a path of length $i-1$ and v_i, v_{i+2}, v_{i+4}, $\ldots, v_{n-2}$, v_n, v_{n-1}, v_{n-3}, $\ldots, v_{i+3}$, v_{i+1} is a path of length $n-i+1$ if n is odd and v_i, v_{i+2}, v_{i+4} $\ldots, v_{n-1}$, v_n, v_{n-2}, v_{n-4}, $\ldots, v_{i+3}$, v_{i+1} is a path of length $n-i+1$ if n is even. Therefore $\delta(v_i, v_{i+1}) = max\{i-1, n-i+1\}$. If i is even, then the path v_i, v_{i-2}, $v_{i-4} \ldots, v_2$, v_1, v_3, v_5, $\ldots$, v_{i-1}, v_{i+1} is of length $i-1$, for n odd, the path v_i, v_{i+2}, v_{i+4}, $\ldots$, v_{n-1}, v_n, v_{n-2}, v_{n-4}, $\ldots$, v_{i+3}, v_{i+1} of length $n-i+1$ and for n even, the path v_i, v_{i+2}, v_{i+4}, $\ldots$, v_{n-2}, v_n, v_{n-1}, v_{n-3}, $\ldots$, v_{i+3}, v_{i+1} is of length $n-i+1$. Therefore in any case $\delta(v_i, v_{i+1}) = max\{i-1, n-i+1\}$.

If $i \neq j, j-1$ then we prove the result for n odd the case n is even follows similarly.

Case 1: Both i and j are odd

In this case both $i-1$ and $n-j$ are even. The path $v_i, v_{i-2}, v_{i-4}, \ldots, v_1$, $v_2, v_4, \ldots, v_{i-1}$, v_{i+1}, v_{i+2}, $v_{i+3}, \ldots, v_{j-1}$, v_{j+1}, $v_{j+3}, \ldots, v_{n-1}$, v_n, v_{n-2}, v_{n-4}, $\ldots, v_j$ is of length $n-1$.

Case 2: Both i and j are even

In this case both $i-1$ and $n-j$ are odd. The path $v_i, v_{i-2}, v_{i-4}, \ldots, v_2$, v_1, v_3, v_5, $\ldots, v_{i-1}$, v_{i+1}, v_{i+2}, $v_{i+3}, \ldots, v_{j-1}$, $v_{j+1}, v_{j+3}, \ldots$, v_{n-2}, v_n, v_{n-1}, $v_{n-3} \ldots, v_{j+2}, v_j$ is of length $n-1$.

Case 3: i is odd and j is even

In this case $i-1$ is even and $n-j$ is odd. The path v_i, v_{i-2}, $v_{i-4} \ldots, v_1$, v_2, v_4, $\ldots, v_{i-1}$, v_{i+1}, v_{i+2}, $v_{i+3}, \ldots, v_{j-1}$, $v_{j+1}, v_{j+3}, \ldots$, v_{n-1}, v_n, v_{n-2}, $v_{n-4}, \ldots, v_{j+2}, v_j$ is of length $n-1$.

Case 4: i be even and j be odd

In this case $i-1$ is odd and $n-j$ is even. The path v_i, v_{i-2}, $v_{i-4}, \ldots, v_2$, v_1, v_3, $v_5, \ldots, v_{i-1}$, v_{i+1}, v_{i+2}, $v_{i+3}, \ldots, v_{j-1}$, $v_{j+1}, v_{j+3}, \ldots$, v_{n-1}, v_n, v_{n-2}, $v_{n-4}, \ldots, v_{j+2}, v_j$ is of length $n-1$.

Thus if $i \neq j-1, j$, we can find a $v_i - v_j$ path of length $n-1$, hence $\delta(v_i, v_j) = n-1$. ☐

Theorem 8.2. *For any integer $n \geq 3$, $\beta_b(P_n^2) = 2$.*

Proof. For $n = 3$ the graph P_n^2 is complete and hence result follows by the Theorem 2.2. Let $n \geq 4$. By the Theorem 2.1, it enough to find a resolving set of cardinality 2. For $n = 4$ the graph $P_n^2 \cong K_4 - e$ and $\beta(K_4 - e) = 2$, hence the result holds in this case. Let $n > 4$. We prove $S = \{v_2, v_3\}$ is a bi-resolving set. For any $i < j$, let $v_i, v_j \in V - S$ such that $d(v_i, v_2) = d(v_j, v_2)$ and $d(v_j, v_2) = d(v_j, v_3)$ then $i = 1$ and $j = 4$. Then by the Lemma 8.1, $\delta(v_1, v_2) = n - 2$ and $\delta(v_4, v_2) = n - 1$ and hence v_2 resolves v_i and v_j. Thus $\beta_b(P_n^2) = 2$. □

Lemma 8.3. *For every integer $n \geq 4$, the length of longest path between any two vertices v_i and v_j of P_n^3 is*

$$\delta(v_i, v_j) = \begin{cases} n & if\ i = j \\ n-1 & if\ i \neq j \end{cases}$$

Proof. We observe that P_n^2 is a subgraph of P_n^3. Therefore by the Lemma 8.1 it is enough to prove the result only in the case $i = j - 1\ or\ j + 1$. Without loss of generality we assume that $i = j - 1$ (other case follows by interchanging the rolls of i and j). If i is odd, then for n even, the path v_i, v_{i-2}, v_{i-4}, …, v_3, v_1, v_2, v_4, …, v_{i-1}, v_{i+2}, v_{i+4}, …, v_{n-1}, v_n, v_{n-2}, v_{n-4}, …, v_j is of length $n - 1$ and for n odd, the path v_i, v_{i-2}, v_{i-4}, …, v_3, v_1, v_2, v_4, …, v_{i-1}, v_{i+2}, v_{i+4}, …, v_{n-2}, v_n, v_{n-1}, v_{n-3}, …, v_j is of length $n - 1$. If i is even, then for n even, the path v_i, v_{i-2}, v_{i-4}, …, v_2, v_1, v_3, v_5, …, v_{i-1}, v_{i+2}, v_{i+4}, …, v_{n-2}, v_n, v_{n-1}, v_{n-3}, …, v_j is of length $n - 1$ and for n odd, the path v_i, v_{i-2}, v_{i-4}, …, v_2, v_1, v_3, v_5, …, v_{i-1}, v_{i+2}, v_{i+4}, …, v_{n-1}, v_n, v_{n-2}, v_{n-4}, …, v_j is of length $n - 1$. Therefore for $i = j - 1$, $\delta(v_i, v_j) = n - 1$. □

Theorem 8.4. *For every integer $n \geq 4$, $\beta_b(P_n^3) = \beta(P_n^3)$.*

Proof. By the Lemma 8.3, we find that whenever $d(v_k, v_i) = d(v_k, v_j)$, then $\delta(v_k, v_i) = \delta(v_k, v_j)$. Therefore a vertex v_k resolves pair of vertices v_i, v_j if and only if $d(v_k, v_i) \neq d(v_k, v_j)$. Hence $\beta_b(P_n^3) = \beta(P_n^3)$. □

Since P_n^3 is a subgraph of P_n^k for all k, $3 < k < n$, By the Lemma 8.3, for any two vertices v_i, v_j in the graph P_n^k, we find

$$\delta(v_i, v_j) = \begin{cases} n & if\ i = j \\ n-1 & if\ i \neq j \end{cases}$$

Hence the following result.

Theorem 8.5. *For any two positive integers n, k with $k < n$, $\beta_b(P_n^k) = \beta(P_n^k)$.*

Lemma 8.6. *For integer $n \geq 4$, the length of longest path between any two vertices v_i and v_j of C_n^2 is*

$$\delta(v_i, v_j) = \begin{cases} n & if\ i = j \\ n-1 & if\ i \neq j \end{cases}$$

Proof. The graph C_n^2 is hamiltonian. Therefore $\delta(v_i, v_i) = n$ for all i. Let v_i and v_j be any two vertices of C_n^2. without loss of generality we assume $i < j$. If $j - i = 1$ then the path $v_j, v_{j+1}, v_{j+2}, \ldots, v_{n-1}, v_n, v_1, v_2, v_3, \ldots, v_{i-1}, v_i$ is a path of length $n - 1$. Therefore for $j - i = 1$, $\delta(v_i, v_j) = n - 1$. If $j - i \geq 3$ and it is odd, then the path $v_j, v_{j+1}, v_{j+2}, \ldots, v_{n-1}, v_n, v_1, v_2, v_3, \ldots, v_{i-1}, v_{i+1}, v_{i+3}, \ldots, v_{j-2}, v_{j-1}, v_{j-3}, v_{j-5}, \ldots, v_{i+2}, v_i$ is of length $n - 1$. If $j - i$ is even, then the path $v_j, v_{j+1}, v_{j+2}, \ldots, v_{n-1}, v_n, v_1, v_2, v_3, \ldots, v_{i-1}, v_{i+1}, v_{i+3}, \ldots, v_{j-1}, v_{j-2}, v_{j-4}, \ldots, v_{i+2}, v_i$ is of length $n - 1$. Thus for $i \neq j$, $\delta(v_i, v_j) = n - 1$. □

In view of the Lemma 8.6, one can conclude that $\beta_b(C_n^2) = \beta(C_n^2)$. Since C_n^2 is a subgraph of C_n^k for each $k \geq 3$, the length of longest path between any two vertices v_i and v_j of C_n^k is given by

$$\delta(v_i, v_j) = \begin{cases} n & if\ i = j \\ n-1 & if\ i \neq j \end{cases}$$

Hence the following theorem.

Theorem 8.7. *For any two positive integers n, k with $k < n$, $\beta_b(C_n^k) = \beta(C_n^k)$.*

9 Conclusion

In this paper we have shown a new way of finding navigators in a network by considering detour of a vertex from navigators. It is important to note that in some cases bi-metric dimension is much less than conventional metric dimension, thereby indicating cost reduction in establishing a network of communication system. We have also invented a novel and easy way of obtaining bi-metric dimension of desired number. Further there is a scope of finding new results.

Many questions remain to be investigated, few of them are listed here.

1. For the given positive integers l, m, with $l \leq m$, Is there a graph G of order n such that $\beta_b(G) = l$ and $\beta(G) = m$? If so how to construct them.
2. Determine the least positive integer k such that $\beta(G) - \beta_b(G) \leq k$ for any graph G.

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Competing Interests

The authors declare that no competing interests exist.

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