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The Modified Galerkin Method for the Modified Wave Equation for the Shape of Superellipsoid

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Abstract

The objective of this work was to find the numerical solution of the Impendence problem for the Helmholtz equation for a smooth superellipsoid. The superellipsoid is a shape that is controlled by two parameters. There are some numerical issues in this type of an analysis; any integration method is affected by the wave number k, because of the oscillatory behavior of the fundamental solution. The Helmholtz equation, which is the modified wave equation, is used in many scattering problems. This project was funded by NASA RI Space Grant for testing of the Robin boundary condition for the shape of the superellipsoid. One practical value of all these computations can be getting a shape for the engine nacelles in a ray tracing the space shuttle. We significantly reduced the number of terms in the infinite series needed to modify the original integral equation and used the Green's theorem to solve the integral equation for the boundary of the surface.

Keywords: Helmholtz equation, galerkin method, superellipsoid.

1 Introduction

The main objective of this paper is to solve the Robin boundary value problem for the Helmholtz equation given by

$$\Delta u + k^2 u = 0, \ \operatorname{Im} k \geq 0, \tag{1.1}$$

where k is the wave number. In this paper we looked at specifically the superellipsoid region, a versatile primitive which is controlled by two parameters. In this case we noted that there are numerical issues with very small and very large parameters. The points we looked at come from all directions as shown in Diagram 1. These results are available for the Helmholtz equation with the Dirichlet boundary condition [1], but not for the impendence condition.

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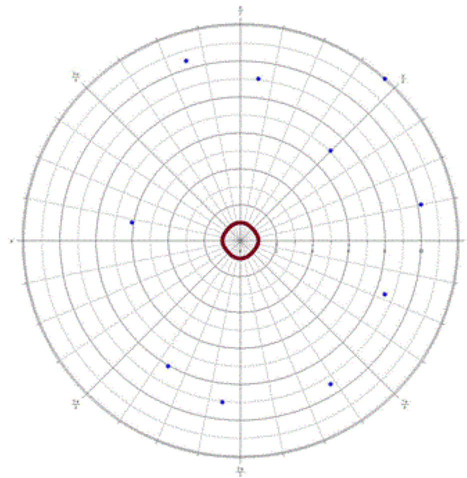

Diagram 1. Cross section of the superellipsoid and the external points.

2 Definitions

It is necessary to develop a method which is uniquely solvable for all frequencies k which is a challenge. Let S be a closed bounded surface in $\Re^3$ and assume it belongs to the class of C^2. Let D_-, D_+, denote the interior and exterior of S respectively. We use Green's theorem as the background for the problem. The exterior Robin problem for the Helmholtz's equation is given by

$$\begin{aligned} \Delta u(A) + k^2 u(A) &= 0, A = (x, y, z) \in D_+, \ \operatorname{Im} k \geq 0 \\ u(p) &= f(p), \ \ p \in S, \end{aligned} \tag{2.1}$$

with f a given function and u satisfying the Sommerfeld radiation condition.

2.1 Framework of the Boundary Value Problems

The exterior Robin problem was written as an integral equation. We represented the solution as a modified single layer potential, based on the modified fundamental solution [2,6,7,8].

$$u(A) = \int_S u(q) \frac{\partial(\frac{e^{ikr}}{4\pi r} + \chi(A,q))}{\partial \nu_q} d\sigma_q \text{with } A \in D_+ \text{ where } r = |A-q|. \tag{2.2}$$

The series of radiating waves is given by

$$\chi(A,q) = ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} h_n^{(1)}(k|A|) Y_n^m (\frac{A}{|A|}) h_n^{(1)}(k|q|) \overline{Y_n^m} (\frac{q}{|q|}). \tag{2.3}$$

This addition of the infinite series to the fundamental solution was in order to remove the singularity that occurs when $A = q$.

Here $h_n^{(1)}$ denote the spherical Hankel function of the first kind and of order n, Y_n^m, $n = -m,...m$ are the linearly independent spherical harmonics of order m.

By letting A tend to a point $p \in S$, we obtained the following integral equation based on the Fredholm equations of the second kind

$$-2\pi\mu(p) + \int_S \mu(q)\frac{\partial\Psi(p,q)}{\partial\nu_q}d\sigma_q = -4\pi f(p), p \in S, \tag{2.4}$$

where

$$\Psi = \frac{-e^{ikr_{qp}}}{r} - 4\pi\chi(p,q).$$

Kleinman and Roach [3] gave an explicit form of the coefficient a_{nm} that minimizes the upper bound on the spectral radius. If B is the exterior of a sphere radius R with center at the origin then the optimal coefficient for the Robin problem was given by [4].

$$a_{nm} = -\frac{1}{2}\left(\frac{j_n(kR)}{h_n^{(1)}(kR)} + \frac{j_n'(kR)}{h_n^{(1)'}(kR)}\right) \text{ for } n = 0,1,2...and\ m = -n,...n. \tag{2.5}$$

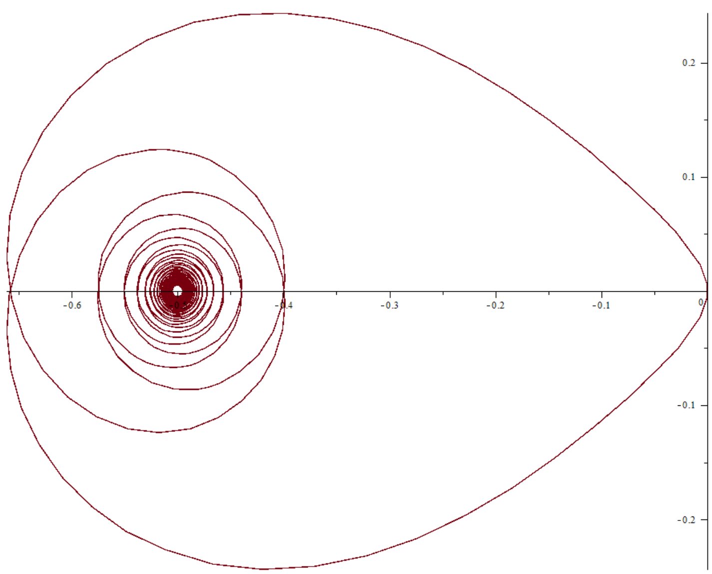

Diagram 2. Coefficient a_{nm}

This choice of the coefficient minimizes the condition number, and as seen in Diagram 2 $-1 < a_{nm} < 0$ for all n. Further the coefficient converges to 0.5 and is applicable for spherical regions [5].

This coefficient choice gave good results for the superellipsoid for the Dirichlet condition [1].

3 Properties of the Integral Operator *K*

We know that the series χ can be differentiated term by term with respect to any of the variables and that the resulting series is uniformly convergent. Also the series χ is a solution to the Helmholtz equation satisfying the Sommerfeld radiation condition for $|x|,|y| > R$, when $B = \{x \;:\; |x| \le R\}$ is contained in D.

By (Theorem 3.5 [2] any two times continuously differentiable solution of the Helmholtz's equation is analytic and analytic functions are infinitely differentiable. So the series $\chi(p,q)$ is infinitely differentiable with respect to any of the variables p,q. Furthermore it is easy to see that if μ is bounded and integrable and $S \in C^l$ then $\int_S \chi(p,q)\mu(q)d\sigma_q \in C^l(S)$ and $\int_S \frac{\partial \chi(p,q)}{\partial \nu_q}\mu(q)d\sigma_q \in C^l(S)$.

4 The Framework

By converting to a new integral equation defined on the unit sphere we could apply the Galerkin Method to this new equation, using spherical polynomials to define the approximating subspaces. Then we obtained the new equation over U,

$$-2\pi\hat{\mu} + \hat{K}\hat{\mu} = -4\pi \;\hat{f}, \hat{f} \in C(U). \tag{4.1}$$

The notation “^” denotes the change of variable from S to U. Galerkin's method for solving (4.1) for the Impendence boundary condition is given by

$$(-2\pi + P_N\hat{K})\widehat{\mu_N} = -4\pi P_N\hat{f}. \tag{4.2}$$

The solution is given by

$$\widehat{\mu_N} = \sum_{j=1}^{d} \alpha_j h_j$$

$$-2\pi\alpha_i(h_1, h_1) + \sum_{j=1}^{d} \alpha_j\left(\hat{K}h_j, h_i\right) = 4\pi\left(\hat{f}, h_i\right), i = 1,...d. \tag{4.3}$$

The convergence of μ_N to μ in $L^2(S)$ is straightforward. We know from previous literature that $P_N \hat{\mu} \to \hat{\mu}$ for all $\hat{\mu} \in L^2(U)$. From standard results it follows that $\left\| \hat{K} - P_N \hat{K} \right\| \to 0$ and we can obtain the desired convergence.

4.1 The Approximation of True Solutions for the Robin Problem

Given μ_N an approximate solution, we defined the approximate solution u_N by

$$u_N(A) = \int_S \mu_N(q) \frac{\partial}{\partial \upsilon_q} \left(\frac{e^{ikr_{qA}}}{4\pi r_{qA}} + \chi(A,q) \right) d\sigma_q, \ A \in D_+ \tag{4.4}$$

To show the convergence of $u_N(A)$, we used the following lemma.

Lemma 1

$$\sup_{A \in K} \int_S \left| \frac{\partial}{\partial \nu_q} \left(\frac{e^{ikr_{qA}}}{4\pi r_{qA}} + \chi(A,q) \right) \right| d\sigma_q < \infty, \tag{4.5}$$

where K is any compact subset of D [9,10].

4.2 Implementation of the Galerkin Method for the Robin Problem

The coefficients $\left(\hat{K}h_j, h_i\right)$ are fourfold integrals with a singular integrand. Because the Galerkin coefficients $\left(\hat{K}h_j, h_i\right)$ depends only on the surface S, we calculated them separately for $N \le N_{\max}$. To decrease the effect of the singularity in computing $\hat{K}h_j(\hat{p})$ in the Robin case, we used the identity

$$\int_S -\frac{\partial}{\partial \nu_q} \frac{1}{r_{qp}} d\sigma_q = 2\pi, \ p \in S \text{ where } r_{qp} = |p - q|. \tag{4.6}$$

5 Numerical Examples/Experimental Surfaces

We used the true solution

$$u_1(x,y,z) = \frac{e^{ikr}}{r} \tag{5.1}$$

for our calculations.

For this analysis it is important to realize that the superellipsoid is simply connected and is of infinite extent.

Table 1. Absolute error for varying interior integration nodes

Points	Absolute errors		
	16, 8	**32, 8**	**32, 16**
(-4,-7,2)	2.42E-04	2.20E-04	2.06E-04
(5,5,5)	2.35E-04	2.16E-04	2.04E-04
(-6,1,7)	2.22E-04	2.06E-04	1.96E-04
(8,-3,5)	2.05E-04	1.88E-04	1.77E-04
(10,2,3)	1.90E-04	1.72E-04	1.61E-04
(-3,10,4)	1.81E-04	1.65E-04	1.54E-04
(8,9,2)	1.66E-04	1.50E-04	1.39E-04
(5,-8,11)	1.41E-04	1.62E-04	1.26E-04
(1,9,12)	1.36E-04	1.27E-04	1.22E-04
(-1,-9,20)	9.34E-05	8.80E-05	8.57E-05

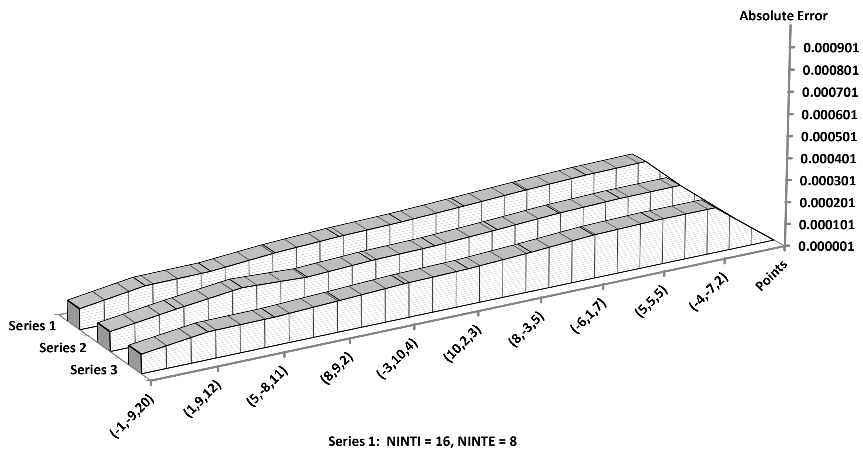

Fig. 1. Further the points from the boundary better the convergence

Table 2. Convergence results for varying wave numbers

k-value / Points	Absolute Errors							
	0.1	**0.5**	**0.9**	**1**	**1.1**	**1.9**	**2.8**	**5.2**
(-4,-7,2)	1.20E-04	1.69E-04	2.29E-04	2.42E-04	2.54E-04	2.84E-04	5.81E-04	1.08E-03
(5,5,5)	1.15E-04	1.60E-04	2.21E-04	2.35E-04	2.49E-04	3.46E-04	4.77E-04	1.91E-03
(-6,1,7)	1.08E-04	1.50E-04	2.08E-04	2.22E-04	2.36E-04	3.74E-04	4.95E-04	1.64E-03
(8,-3,5)	1.01E-04	1.40E-04	1.93E-04	2.05E-04	2.16E-04	2.86E-04	4.20E-04	1.36E-03
(10,2,3)	9.36E-05	1.31E-04	9.44E-05	1.90E-04	2.00E-04	2.28E-04	4.37E-04	7.83E-04
(-3,10,4)	8.90E-05	1.22E-04	1.71E-04	1.81E-04	1.91E-04	2.27E-04	3.96E-04	7.18E-04
(8,9,2)	8.15E-05	1.14E-04	1.57E-04	1.66E-04	1.74E-04	1.88E-04	4.01E-04	9.34E-04
(5,-8,11)	6.88E-05	9.56E-05	1.32E-04	1.41E-04	1.50E-04	2.40E-04	3.28E-04	1.03E-03
(1,9,12)	6.63E-05	9.20E-05	1.28E-04	1.36E-04	1.45E-04	2.39E-04	3.26E-04	7.97E-04
(-1,-9,20)	4.54E-05	6.27E-05	8.71E-05	9.34E-05	1.00E-04	1.78E-04	2.39E-04	2.98E-04

From the above Table 1 and Fig. 1, we see that for the points away from the boundary there is much greater accuracy than for points near the boundary. This is because the integrand is more singular at points near the boundary. Also the convergence results are much better for small values of k.

From the above Table 2 and Fig. 2 we see that increasing the number of Galerkin coefficients give a better result. This is due to the following fact: the kernel function involves $\sin kr$ and $\cos kr$, these trigonometric functions are much more oscillatory when k becomes large. Therefore in this case we must increase the integration nodes to achieve the same accuracy.

Remark:
We chose more interior nodes because the integrand of $(h_i, \widehat{K}h_j)$ is smoother than the integrand of $\widehat{K}h_j$.

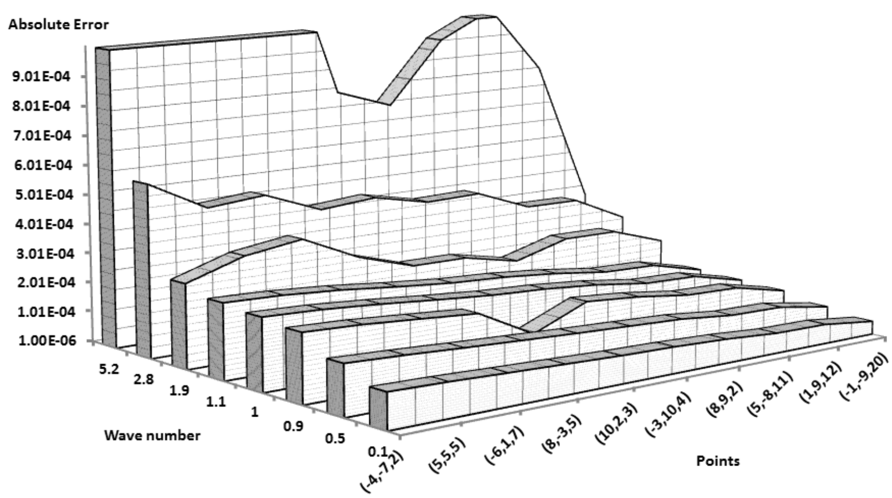

Fig. 2. The best convergence results were for smaller wave numbers

Table 3. The absolute errors for a range of superellipsoids ranging from [0.5 – 1.8]

	Absolute errors					
n-value / Points	**0.5**	**0.9**	**1.2**	**1.4**	**1.7**	**1.8**
(-4,-7,2)	6.48E-03	3.23E-04	1.34E-04	2.42E-04	6.27E-04	8.03E-04
(5,5,5)	4.91E-03	2.63E-04	1.39E-04	2.35E-04	6.29E-04	8.18E-04
(-6,1,7)	3.63E-03	2.09E-04	1.38E-04	2.22E-04	6.10E-04	8.03E-04
(8,-3,5)	4.61E-03	2.41E-04	1.19E-04	2.05E-04	5.43E-04	7.03E-04
(10,2,3)	5.01E-03	2.49E-04	1.06E-04	1.90E-04	4.91E-04	6.29E-04
(-3,10,4)	4.57E-03	2.30E-04	1.02E-04	1.81E-04	4.70E-04	6.04E-04
(8,9,2)	4.57E-03	2.24E-04	9.23E-05	1.66E-04	4.23E-04	5.40E-04
(5,-8,11)	2.25E-03	1.34E-04	8.75E-05	1.41E-04	3.91E-04	5.16E-04
(1,9,12)	2.02E-03	1.23E-04	8.56E-05	1.36E-04	3.81E-04	5.04E-04
(-1,-9,20)	1.05E-03	7.37E-05	6.07E-05	9.34E-05	2.70E-04	3.61E-04

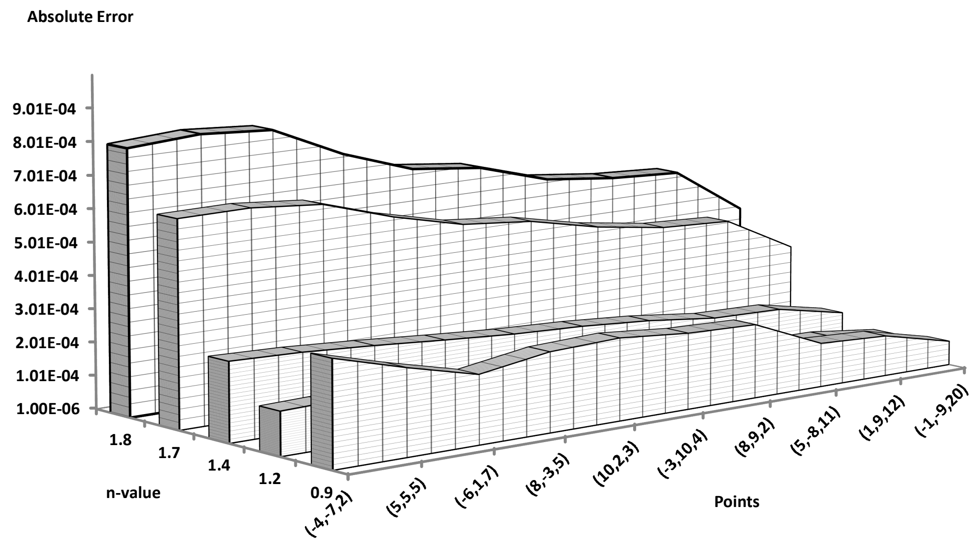

Fig. 3. Convergence result for varying n values. The best result was for the superellipsoid with n = 1.4

When n is changed, see Fig. 3 and above Table 3, we obtained new shapes of superellipsoids. The optimal results were given when n is between 1.2 and 1.4.

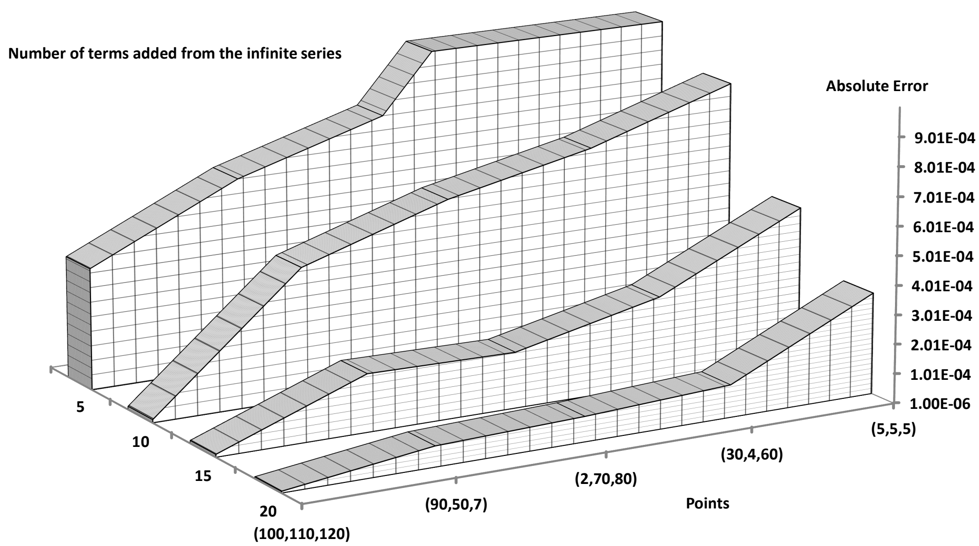

Fig. 4. More terms better the convergence

Table 4. Absolute error for varying number of terms from the infinite series

	Absolute errors			
Terms / Points	5	10	15	20
(5,5,5)	8.96E-03	8.71E-04	5.35E-04	3.45E-04
(30,4,60)	1.38E-03	7.14E-04	3.02E-04	1.11E-04
(2,70,80)	8.00E-04	6.08E-04	1.91E-04	8.62E-05
(90,50,7)	6.51E-04	4.49E-04	1.98E-04	7.10E-05
(100,110,120)	4.16E-04	1.20E-05	1.06E-05	6.66E-06

The above Fig. 4 and Table 4 shows that further the points are from the boundary the better the convergence results are for this problem, which is the expected result.

6 Conclusion

We allowed only a finite number of the coefficients a_{nm} to be different from zero. According [11] this is sufficient to ensure uniqueness for the modified integral equation in a finite range of wave numbers k. In practical applications, one is usually concerned with a finite range of k. From the above examples, we see that the error is effected by the boundary S, interior nodes, exterior nodes, k and n. If we want to obtain more accuracy, we must increase the number of integration nodes for calculating the Galerkin coefficients $(\widehat{K}h_j, h_i)$. Some of the increased cost comes from the complex number calculations, which is an intrinsic property of the Helmholtz equation [12,13,14].

Furthermore any integration method is affected by k, due to the oscillatory behavior of the fundamental solution $\frac{e^{ikr}}{r}$. Thus the superellipsoid shape is a viable shape for testing incoming waves that absorb and reflect at the boundary (Robin condition) for smooth simply connected surfaces.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Warnapala Y, Dinh H. Numerical Solution of the Helmholtz's Equation for the superellipsoid via the Galerkin Method, Communications in Numerical Analysis. 2013;1-12.

[2] Colton David, Kress Rainer. Integral equation methods in scattering theory; 1983.

[3] Kleinman RE, Roach GF. On modified Green functions in exterior problems for the Helmholtz equation. Proc. R. Soc Lond. 1982;383:313-33.

[4] Warnapala Y, Morgan E. The numerical solution of the exterior dirichlet problem for helmholtz's equation via modified green's functions approach for the oval of cassini. Far East Journal of Applied Mathematics. 2009;34:1-20.

[5] Jones DS. Integral equations for the exterior acoustic problem. Q J Mech Appl Math. 1974;27:129-142.

[6] Burton AJ. The solution of Helmholtz' equation in exterior domains using integral equations. Division of Numerical Analysis and Computing, Teddington, Middlesex, NPL report NAC 30; 1973.

[7] Atkinson KE. The numerical solution of the Laplace's equation in three dimensions. SIAM J Numer Anal. 1982;19:263-274.

[8] Atkinson KE. The numerical solution of integral equations of the second kind. Cambridge University Press; 1998.

[9] Jones DS. Integral equations for the exterior acoustic problem. Q J Mech Appl Math. 1974;27:129-142.

[10] Jost G. Integral equations with modified fundamental solution in time-harmonic electromagnetic scattering. IMA Journal of Appl Math. 1988;40:129-143.

[11] Kleinman RE, Roach GF. Operators of minimal norm via modified Green's functions. Proceedings of the Royal Society of Edinburgh. 1983;94:163-178.

[12] Lin TC. The numerical solution of Helmholtz equation using integral equations. Ph.D. Thesis, University of Iowa, Iowa City, Iowa; 1982.

[13] Schenk, Harry A. Improved integral formulation for acoustic radiation problems. Journal of the Acoustical Society of America. 1967;41-58.

[14] Ursell F. On the exterior problems of acoustics. II. Proc Cambridge Phil Soc. 1978;84:545-548.

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