British Journal of Mathematics & Computer Science
4(18): 2665-2675, 2014

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

The Exact Distribution of The Ratio of Two Independent Hypoexponential Random Variables

Therrar Kadri[1*] and Khaled Smaili[2]

[1] *Department of Mathematics, Faculty of Sciences, Beirut Arab University, Beirut, Lebanon.*
[2] *Department of Applied Mathematics, Faculty of Sciences, Lebanese University, Zahle, Lebanon.*

Original Research Article

Received: 03 March 2014
Accepted: 11 June 2014
Published: 08 July 2014

Abstract

The distribution of ratio of two random variables are of interest in many areas of the sciences. The study of the ratio of same family and finding their exact density expression was examined by many authors for decades. Some are solved and others gave approximations, but many still unsolved. In this paper, we consider the ratio of two independent Hypoexponential distributions. We find the exact expressions for the probability density function, the cumulative distribution function, moment generating function, the reliability function and hazard function, which was proved to be a linear combination of the Generalized-F distribution.

Keywords: Ratio Distribution, Hypoexponential Distribution; Erlang Distribution; Generalized-F Distribution; Log-logistic Distribution; Probability Density Function; Cumulative Distribution Function; Reliability Function; Hazard Function.

1 Introduction

The distributions of ratio of random variables are widely used in many applied problems of engineering, physics, number theory, order statistics, economics, biology, genetics, medicine, hydrology, psychology, classification, and ranking and selection, see [1, 2, 3, 4]. Examples include safety factor in engineering, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, inventory ratios in economics and Mendelian inheritance ratios in genetics, see [1, 2]. Also ratio distribution involving two Gaussian random variables are used in computing error and outage probabilities, see [5]. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures and the aging of

*Corresponding author: E-mail: therrar@hotmail.com

concrete pressure vessels, see [6] and [7]. An important example of ratios of random variables is the stress strength model in the context of reliability. It describes the life of a component which has a random strength Y and is subjected to random stress X. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $Y > X$. Thus, $Pr(X < Y)$ is a measure of component reliability see [6] and [7].

The ratio distribution X/Y have been studied by several authors especially when X and Y are independent random variables and come from the same family. For a historical review, see the papers by Marsaglia [8] and Korhonen and Narula [9] for the normal family, Press [10] for Student's t family, Basu and Lochner [11] for the Weibull family, Hawkins and Han [12] for the non-central chi-squared family, Provost [13] for the gamma family, Pham-Gia [14] for the beta family, Nadarajah and Gupta [15] for the Logistic family, Nadarajah and Kotz [16] for the Frèchet family, Ali, Pal, and Woo [1] for the inverted gamma family, Nadarajah [17] for Laplace family, and Pham-Gia and Turkkan [7] for the Generalized-F family. However, there is no work done when X and Y are two independent Hypoexponential random variables for their general case. The particular case of the Hypoexponential distribution, the Erlang distribution, was solved knowing that the Erlang distribution is a particular case of the gamma distribution and that latter was examined by Provost [13] and Carlos in [18]. They showed that the ratio of two gamma distributions is the Generalized-F distribution. Finally, if X and Y are two independent Exponential distributions, the simple case of the Hypoexponential distribution, then X/Y is a particular case of log-logistic distribution.

In our paper, we consider the ratio of two independent Hypoexponential random variables. We examine the general case of this problem when the stages of the Hypoexponential distribution do not have to be distinct. We use the expression of the probability density function (PDF) for the general case of the Hypoexponential distribution given by Smaili et al. in [19]. The expression of the PDF was written as a linear combination of the PDF of the Erlang distribution. We start by finding a more direct expression of the coefficients of this linear combination. In this manner, we solve the general problem of X/Y and derive exact expressions of the PDF, cumulative distribution function (CDF), reliability function, hazard function and the moment generating function. We showed that the PDF of X/Y is a linear combination of the Generalized-F distribution. Next, we give -in particular - the expression of PDF and CDF of the ratio when X and Y are independent Hypoexponential distribution of different stages. Eventually, we verify the two particular cases, the Erlang and the Exponential distributions.

2 Some Preliminaries

2.1 The Hypoexponential distribution

The Hypoexponential distribution is the distribution of the sum of $m \geq 2$ independent Exponential random variables. The general case of the Hypoexponential distribution is when the m exponential stages do not have to be distinct. This general case can be written as $S_m = \sum_{i=1}^{n}\sum_{j=1}^{k_i} X_{ij}$, where X_{ij} is an Exponential random variable with parameter α_i, $i = 1, 2, ..., n$, and written as $S_m \sim Hypoexp\left(\overrightarrow{\alpha}, \overrightarrow{k}\right)$ of parameters $\overrightarrow{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\overrightarrow{k} = (k_1, k_2, ..., k_n)$ with $m = \sum_{i=1}^{n} k_i$, see [19]. Smaili et al. in [19] gave a modified expression for the PDF of the Hypoexponential random variable as

$$f_{S_m}(t) = \sum_{i=1}^{n}\sum_{j=1}^{k_i} A_{ij} f_{E_{ij}}(t) \tag{2.1}$$

where $f_{E_{ij}}(t) = \frac{(\alpha_i t)^{j-1}\alpha_i e^{-\alpha_i t}}{(j-1)!}$ if $t > 0$ and $f_{E_{ij}}(t) = 0$ if $t \leq 0$, is the PDF of the Erlang distribution E_{ij} with j the shape parameter and α_i the rate parameters, written as $E_{ij} \sim Erl(j, \alpha_i)$

for $1 \leq i \leq n$, $1 \leq j \leq k_i$. They showed that

$$\sum_{i=1}^{n}\sum_{j=1}^{k_i} A_{ij} = 1 \tag{2.2}$$

and gave an expression of A_{ij} in a recursive way, stated in the following Lemma.

Lemma 1. *Let* $1 \leq i \leq n$*,* $1 \leq j \leq k_i$*. Then* $A_{ij} = \frac{\prod_{i=1}^{n}\alpha_i^{k_i}}{\alpha_i^{j}(k_i-j)!}\lim_{s\to-\alpha_i} g_i^{(k_i-j)}(s)$*, where* $g_i(s) = \prod_{j=1,j\neq i}^{n}\frac{1}{(s+\alpha_j)^{k_j}}$ *and* $g_i^{(q)}(s)$*, the* q^{th} *derivative of* $g_i(s)$*, is obtained iteratively as* $g_i^{(q)}(s) = \sum_{l=0}^{q-1}[\binom{q-1}{l}(-1)^{l+1}\left(\sum_{j=1,j\neq i}^{n}\frac{(l)!k_j}{(s+\alpha_j)^{l+1}}\right)g_i^{(q-l-1)}(s)]$ *for* $q \geq 1$ *and* $g_i^{(0)}(s) = g_i(s)$.

This coefficient of the linear combination is used throughout our paper. So we worked on modifying the expression in a more simple form for computation, given in the following proposition.

Proposition 1. *Let* $1 \leq i \leq n$*,* $1 \leq j \leq k_i$*. Then* $A_{i,k_i} = \prod_{j=1,j\neq i}^{n}\left(1-\frac{\alpha_i}{\alpha_j}\right)^{-k_j}$ *and for* $j = k_i - 1$*, down to* 1*.* $A_{ij} = \frac{1}{k_i-j}\sum_{l=1}^{k_i-j}[\left(\sum_{p=1,p\neq i}^{n}k_p(1-\frac{\alpha_p}{\alpha_i})^{-l}\right)A_{i,j+l}]$

Proof. Taking the formulas in Lemma 1, $j = k_i$, we get $A_{i,k_i} = \frac{\prod_{i=1}^{n}\alpha_i^{k_i}}{\alpha_i^{k_i}} \times g_i(-\alpha_i) = \frac{\prod_{i=1}^{n}\alpha_i^{k_i}}{\alpha_i^{k_i}} \times \prod_{j=1,j\neq i}^{n}\frac{1}{(-\alpha_i+\alpha_j)^{k_j}} = \prod_{j=1,j\neq i}^{n}\frac{1}{\left(1-\frac{\alpha_i}{\alpha_j}\right)^{k_j}}$. Now, for $1 \leq j \leq k_i - 1$ the expression of A_{ij} in Lemma 1 can be written as $A_{ij} =$

$$\frac{\prod_{i=1}^{n}\alpha_i^{k_i}}{\alpha_i^{j}(k_i-j)!}\sum_{l=0}^{k_i-j-1}[\binom{k_i-j-1}{l}(-1)^{l+1}\left(\sum_{j=1,j\neq i}^{n}\frac{(l)!k_j}{(-\alpha_i+\alpha_j)^{l+1}}\right)g_i^{(k_i-j-l-1)}(-\alpha_i)]$$

$= \frac{\prod_{i=1}^{n}\alpha_i^{k_i}}{\alpha_i^{j}(k_i-j)!}\sum_{l=0}^{k_i-j-1}[\binom{k_i-j-1}{l}(-1)^{l+1}\left(\sum_{j=1,j\neq i}^{n}\frac{(l)!k_j}{(-\alpha_i+\alpha_j)^{l+1}}\right)\frac{(k_i-j-l-1)!\alpha_i^{j+l+1}A_{i,j+l+1}}{\prod_{i=1}^{n}\alpha_i^{k_i}}]$. Therefore, $A_{ij} = \frac{1}{k_i-j}\sum_{l=0}^{k_i-j-1}[\left(\sum_{p=1,p\neq i}^{n}\frac{k_p}{(1-\frac{\alpha_p}{\alpha_i})^{l+1}}\right)A_{i,j+l+1}]$. By replacing the argument $l+1$ by l, we obtain our formula. ☐

A particular case of this distribution is when the m stages are different. This case is written as $X \sim Hypoexp(\overrightarrow{\alpha})$, where $\overrightarrow{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\overrightarrow{k} = (1, 1, ..., 1)$, see [20]. From Eq (2.1) we have the PDF is $f_{S_m}(t) = \sum_{i=1}^{n}\sum_{j=1}^{k_i} A_{ij}f_{E_{ij}}(t) = \sum_{i=1}^{n} A_{i1}f_{E_{i1}}(t)$. However, E_{i1} is the Exponential random variable with parameter α_i and $A_{i1} = \prod_{j=1,j\neq i}^{n}(\frac{\alpha_j}{\alpha_j-\alpha_i}) \triangleq A_i$, for $1 \leq i \leq n$, see [19]. Then the PDF is

$$f_{S_m}(t) = \sum_{i=1}^{n} A_i\alpha_i e^{-\alpha_i t} \text{ for } t > 0 \tag{2.3}$$

Another particular case is when the m stages are identical and the distribution is the Erlang distribution. Taking in Eq (2.1) $\overrightarrow{\alpha} = (\alpha)$, $\overrightarrow{k} = (k)$ and $n = 1$, we write $X \sim Hypoexp(\alpha, k) = Erl(k, \alpha)$, see [19]. Moreover, in [19], they showed that $A_{1j} = 0$ for $1 \leq j \leq k-1$ and $A_{1,k} = 1$.

2.2 Generalized-F distribution

Let X be a random variable that has the Generalized-F distribution or called the generalized beta-prime distribution with three positive parameters v_1, v_2 and γ. The PDF of X is $f(t, v_1, v_2, \gamma) = \frac{\gamma^{v_2} t^{v_1-1}}{B(v_1,v_2)(t+\gamma)^{v_1+v_2}}$, $t \geq 0$, where

$$B(v_1, v_2) = \int_0^1 t^{v_1-1} (1-t)^{v_2-1} \, dt \tag{2.4}$$

is the usual Beta function, see [7]. Taking $\gamma = \frac{\beta}{\alpha}$, where $\alpha > 0$ and $\beta > 0$, we can write the PDF as

$$f(t, v_1, v_2, \frac{\beta}{\alpha}) = \frac{\left(\frac{\beta}{\alpha}\right)^{v_2} t^{v_1-1}}{B(v_1, v_2)\left(t + \frac{\beta}{\alpha}\right)^{v_1+v_2}} \tag{2.5}$$

The Generalized-F distribution is related to the F-distribution with two parameters, where γX is a F distribution with 2α and 2β degrees of freedom, see [7] and [21]. Thus, the distribution of Genralized-F distribution can be obtained from this relation, see [21], [22] and [23]. As a result we have the cumulative distribution function of X is

$$F(x) = I_{\frac{x}{x+\gamma}}\left(v_1, v_2\right), \tag{2.6}$$

where $I_x(a,b) = \frac{B(x,a,b)}{B(a,b)}$ is the regularized incomplete beta function and $B\left(x,a,b\right) = \int_0^x t^{a-1}\left(1-t\right)^{b-1} dt$ is the incomplete beta function. The moment generating function of X is

$$\Phi(t) = \frac{\Gamma\left(v_1+v_2\right)}{\Gamma\left(v_2\right)} U\left(v_1, 1-v_2, -\gamma t\right), \tag{2.7}$$

where $U\left(a,b,z\right) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt} t^{a-1}\left(1+t\right)^{b-a-1} dt$ is the confluent hypergeometric function of the second kind.

The particular case of the Generalized-F distribution is when $v_1 = v_2 = 1$, which is the log-logistic distribution with parameter $1/\gamma$ and 1, we write $X \sim GF(1,1,\gamma) = loglogistic(\frac{1}{\gamma}, 1)$ when $v_1 = v_2 = 1$. In this case we have the PDF of X is

$$f(t) = \frac{\gamma}{\left(t+\gamma\right)^2} \tag{2.8}$$

if $t > 0$ and $f(t) = 0$ if $t \leq 0$, and CDF is

$$F(x) = \frac{x}{x+\gamma} \tag{2.9}$$

if $x > 0$ and $F(x) = 0$ if $x \leq 0$.

3 Ratio of Two Independent Hypoexponential Distributions

In this section, we give the exact expression of the PDF of the ratio of two independent Hypoexponential distributions. The expression is given in the case of the general case of any two independent Hypoexponential distributions. The obtained form of the PDF is a linear combination of the known Generalized-F distribution. As a consequence, the CDF, the reliability function, the hazard function, and the moment generating function of the ratio distribution are given. Considering the particular case of this distribution, the Hypoexponential distribution with different stages, we give a more particular form of the PDF and CDF for the ratio distribution. Next, we apply the two particular cases (1) ratio of two independent Erlang random variables, (2) ratio of two independent Exponential random variables, to our general form of PDF to verify the results obtained in [13] and [18].

3.1 General Case

We suppose that X and Y follow the general case of the Hypoexponential distribution that are independent. We take $X \sim Hypoexp(\overrightarrow{\alpha}, \overrightarrow{k})$, $\overrightarrow{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\overrightarrow{k} = (k_1, k_2, ..., k_n)$ and $Y \sim Hypoexp(\overrightarrow{\beta}, \overrightarrow{l})$, $\overrightarrow{\beta} = (\beta_1, \beta_2, ..., \beta_r)$, $\overrightarrow{l} = (l_1, l_2, ..., l_r)$. From Eq (2.1) the PDFs of X and Y are

$$f_X(t) = \sum_{i=1}^{n}\sum_{j=1}^{k_i} A_{ij} f_{E_{\alpha_{i,j}}}(t) \text{ and } f_Y(t) = \sum_{p=1}^{r}\sum_{q=1}^{l_p} B_{pq} f_{E_{\beta_{p,q}}}(t) \tag{3.1}$$

respectively, where $E_{\omega,\delta} \sim Erl(\delta, \omega)$ and A_{ij} and B_{pq} are given in Proposition 1 for $1 \leq i \leq n$, $1 \leq j \leq k_i$, $1 \leq p \leq r$, and $1 \leq q \leq l_p$.

Theorem 1. *Let X and Y are two independent random variables distribution according to (3.1). Then the PDF of $Z = X/Y$ is given by*

$$f_Z(t) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\sum_{p=1}^{r}\sum_{q=1}^{l_p} A_{ij} B_{pq} f_{W_{j,q,i,p}}(t), \tag{3.2}$$

where $f_{W_{j,q,i,p}}(t)$ is the PDF of the Generalized-F distribution $W_{j,q,i,p} \sim GF(j, q, \frac{\beta_p}{\alpha_i})$.

Proof. The PDF of $Z = X/Y$ is $f_Z(t) = \int_0^\infty y f_X(yt) f_Y(y) dy$ for $t \geq 0$ and since the PDF of X and Y, are given in Eq (3.1), we obtain that

$$f_Z(t) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\sum_{p=1}^{r}\sum_{q=1}^{l_p} A_{ij} B_{pq} \int_0^\infty y \left(f_{E_{\alpha_{i,j}}}(yt) f_{E_{\beta_{p,q}}}(y) \right) dy$$

Next, we compute the product in the above integral by multiplying the PDF of the Erlang distribution corresponding to their parameters, we get $f_{E_{\alpha_{i,j}}}(yt) f_{E_{\beta_{p,q}}}(y) = \frac{\left(\frac{\beta_p}{\alpha_i}\right)^q t^{j-1}}{(j-1)!(q-1)!} y^{q+j-2} e^{-y(t+\frac{\beta_p}{\alpha_i})}$ for $t \geq 0$ and $f_Z(t) = 0$ for $t < 0$. Then for $t \geq 0$,

$$f_Z(t) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\sum_{p=1}^{r}\sum_{q=1}^{l_p} B_{pq} A_{ij} \frac{\left(\frac{\beta_p}{\alpha_i}\right)^q t^{j-1}}{(j-1)!\,(q-1)!} \int_0^\infty y^{q+j-1} e^{-y(t+\frac{\beta_p}{\alpha_i})} dt$$

To evaluate the above integral, we may use the integral $\int_0^\infty x^a e^{-xb} dx = \frac{a!}{b^{a+1}}$ for $b > 0$ and any positive integer a. Thus $\int_0^\infty y^{q+j-1} e^{-y(t+\frac{\beta_p}{\alpha_i})} dy = \frac{(j+q-1)!}{(t+\frac{\beta_p}{\alpha_i})^{j+q}}$. We obtain that

$$f_Z(t) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\sum_{p=1}^{r}\sum_{q=1}^{l_p} A_{ij} B_{pq} \frac{\left(\frac{\beta_p}{\alpha_i}\right)^q (j+q-1)! t^{j-1}}{(j-1)!\,(q-1)!(t+\frac{\beta_p}{\alpha_i})^{j+q}}$$

However, $\frac{(j-1)!(q-1)!}{(j+q-1)!}$ is the particular case of the beta function $B(j, q)$ in equation (2.4), having j and q as integers. We can see that the expression in the above summation is the PDF of the Generalized-F distribution given in Eq (2.5) as $GF(j, q, \frac{\beta_p}{\alpha_i})$. Writing $W_{j,q,i,p} \sim GF(j, q, \frac{\beta_p}{\alpha_i})$, we obtain the result. □

The PDF of the ratio of two independent Hypoexponential distributions, given in Theorem 1, is a linear combination of the Generalized-F distribution. From this linear form, we can find other related functions such as the CDF and moment generating function. Then the CDF of X/Y is a linear combination of the CDF of the Generalized-F distribution. In a similar manner, the moment generating function of X/Y is also a linear combination of the moment generating function of the Generalized-F distribution. The CDF and moment generating function of the Generalized-F distribution are given in (2.6) and (2.7) respectively. Thus, we obtain the CDF and moment generating function, stated in the following two corollaries.

Corollary 1. *Let X and Y be two independent random variables distributed according to (3.1). Then the CDF of $Z = X/Y$ is given by*

$$F_Z(x) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\sum_{p=1}^{r}\sum_{q=1}^{l_p} A_{ij}B_{pq}I_{\frac{x}{x+\frac{\beta_p}{\alpha_i}}}(j,q)$$

where $I_x(a,b)$ is the regularized incomplete beta function.

Corollary 2. *Let X and Y be two independent random variables distributed according to (3.1). Then the moment generating function of $Z = X/Y$ is given by*

$$\Phi_Z(t) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\sum_{p=1}^{r}\sum_{q=1}^{l_p} A_{ij}B_{pq}\Phi_{W_{j,q,i,p}}(t)$$

where $\Phi_{W_{j,q,i,p}}(t) = \frac{\Gamma(j+q)}{\Gamma(q)}U\left(j, 1-q, -\frac{\beta_p t}{\alpha_i}\right)$ and $U(a,b,z)$ is the confluent hypergeometric function of the second kind.

In the next corollary, we give the expressions of the reliability function (survival function) and hazard function (failure rate) of X/Y. The expressions are directly obtained from the defintion of these functions, where the reliability function is $R_Z(x) = 1 - F_Z(x)$ and hazard function is $h_Z(t) = \frac{f_Z(t)}{R_Z(t)}$.

Corollary 3. *Let X and Y be two independent random variables distributed according to (3.1). Then the reliability function of $Z = X/Y$ is*

$$R_Z(x) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\sum_{p=1}^{r}\sum_{q=1}^{l_p} A_{ij}B_{pq}I_{\frac{\gamma_{ip}}{x+\gamma_{ip}}}(q,j)$$

and the hazard function of $Z = X/Y$ is given by

$$h_Z(t) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{k_i}\sum_{p=1}^{r}\sum_{q=1}^{l_p} A_{ij}B_{pq}f_{W_{j,q,i,p}}(t),}{\sum_{i=1}^{n}\sum_{j=1}^{k_i}\sum_{p=1}^{r}\sum_{q=1}^{l_p} A_{ij}B_{pq}I_{\frac{\gamma_{ip}}{x+\gamma_{ip}}}(q,j)}$$

where $I_x(a,b)$ is the regularized incomplete beta function and $\gamma_{ip} = \frac{\beta_p}{\alpha_i}$, $1 \le i \le n$ and $1 \le p \le r$ and $f_{W_{j,q,i,p}}(t)$ is the PDF of the Generalized-F distribution $W_{j,q,i,p} \sim GF(j,q,\frac{\beta_p}{\alpha_i})$.

Proof. We have the $R_Z(x) = 1 - F_Z(x)$, and $F_Z(x)$ is given in Corollary 1 as $F_Z(x) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\sum_{p=1}^{r}\sum_{q=1}^{l_p} A_{ij}B_{pq}I_{\frac{x}{x+\gamma_{ip}}}(j,q)$. However, from [24], we have $I_{\frac{x}{x+\gamma_{ip}}}(j,q) = 1 - I_{1-\frac{x}{x+\gamma_{ip}}}(q,j) = 1 - I_{\frac{\gamma_{ip}}{x+\gamma_{ip}}}(q,j)$. Then we have $F_Z(x) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\sum_{p=1}^{r}\sum_{q=1}^{l_p} A_{ij}B_{pq} - \sum_{i=1}^{n}\sum_{j=1}^{k_i}\sum_{p=1}^{r}\sum_{q=1}^{l_p} A_{ij}B_{pq}I_{\frac{\gamma_{ip}}{x+\gamma_{ip}}}(q,j)$, and from Eq (2.2), the first summation is 1. This gives the above expression of $R_Z(x)$. ☐

3.2 Particular cases

Next, we consider the particular case of the Hypoexponential distribution when the stages are different. Let X and Y be two independent Hypoexponential random variables each distribution of different stages. Thus, $X \sim Hypoexp(\overrightarrow{\alpha})$, $\overrightarrow{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $Y \sim Hypoexp(\overrightarrow{\beta})$, $\overrightarrow{\beta} = (\beta_1, \beta_2, ..., \beta_r)$, and then from Eq (2.3) the PDF of X and Y are

$$f_X(t) = \sum_{i=1}^{n} A_i \alpha_i e^{-\alpha_i t} \text{ and} f_Y(t) = \sum_{p=1}^{r} B_p \beta_p e^{-\beta_p t} \tag{3.3}$$

respectively with $A_i = \prod_{j=1, j\neq i}^{n} (\frac{\alpha_j}{\alpha_j - \alpha_i})$ and $B_p = \prod_{j=1, j\neq p}^{n} (\frac{\beta_j}{\beta_j - \beta_p})$.

Theorem 2. *Let X and Y be two independent random variables with densities given in (3.3). Then the PDF of $Z = X/Y$ is given by*

$$f_Z(t) = \sum_{i=1}^{n} \sum_{p=1}^{r} \frac{A_i B_p \frac{\beta_p}{\alpha_i}}{\left(t + \frac{\beta_p}{\alpha_i}\right)^2}$$

and the CDF is

$$F_Z(x) = 1 - \sum_{i=1}^{n} \sum_{p=1}^{r} \frac{A_i B_p \left(\frac{\beta_p}{\alpha_i}\right)}{x + \frac{\beta_p}{\alpha_i}}.$$

Note that the PDF of X/Y is a linear combination of the PDF of the particular case of log-logistic distribution, $loglogistic(\frac{\alpha_i}{\beta_p}, 1)$ with PDF $f_L(t) = \frac{\frac{\beta_p}{\alpha_i}}{\left(t+\frac{\beta_p}{\alpha_i}\right)^2}$ given in Eq(2.8).

Proof. From Theorem 1, we take the particular cases of the two Hypoexponential distributions when their stages are different. Then $\overrightarrow{k} = (1, 1, ..., 1)$ and $\overrightarrow{l} = (1, 1, ..., 1)$, this gives in the summation in Eq (3.2) the only values are $j = q = 1$. Then the Generalized-F distribution $W_{j,q,i,p}$ is written as $W_{1,1,i,p} \sim GF(1, 1, \frac{\beta_p}{\alpha_i})$. However, from Eq (2.5)

$$f_{W_{1,1,i,p}}(t) = f(t, 1, 1, \frac{\beta_p}{\alpha_i}) = \frac{\frac{\beta_p}{\alpha_i}}{B(1,1)\left(\frac{\beta_p}{\alpha_i} + t\right)^2} = \frac{\frac{\beta_p}{\alpha_i}}{\left(t + \frac{\beta_p}{\alpha_i}\right)^2}$$

Moreover, knowing that in this particular case $B_{pq} = B_p$ and $A_{ij} = A_i$, we obtain the given form of PDF of Z as $f_Z(t) = \sum\limits_{i=1}^{n} \sum\limits_{p=1}^{r} \frac{A_i B_p \left(\frac{\beta_p}{\alpha_i}\right)}{\left(t+\frac{\beta_p}{\alpha_i}\right)^2} = \sum\limits_{i=1}^{n} \sum\limits_{p=1}^{r} A_i B_p f_L(t)$. On the other hand, we conclude the CDF from this linear form by integrating from 0 to x, we obtain that the CDF of Z is $F_Z(x) = \sum\limits_{i=1}^{n} \sum\limits_{p=1}^{r} A_i B_p F_L(x)$, where $F_L(x)$ is the CDF of $f_L(t)$. We have from Eq (2.9), $F_L(x) = \frac{x}{x+\frac{\beta_p}{\alpha_i}}$. Then $F_Z(x) = \sum\limits_{i=1}^{n} \sum\limits_{p=1}^{r} \frac{A_i B_p{}_i x}{\frac{\beta_p}{\alpha_i} + x} = \sum\limits_{i=1}^{n} \sum\limits_{p=1}^{r} \left(A_i B_p - \frac{A_i B_p \left(\frac{\beta_p}{\alpha_i}\right)}{x+\frac{\beta_p}{\alpha_i}}\right)$. However, from Eq (2.2) and by taking the particular cases A_i and B_p, we get that $\sum\limits_{i=1}^{n} A_i = \sum\limits_{p=1}^{r} B_p = 1$. Thus $\sum\limits_{i=1}^{n} \sum\limits_{p=1}^{r} A_i B_p = \sum\limits_{i=1}^{n} A_i \times \sum\limits_{p=1}^{r} B_p = 1$. Thus, we obtained that $F_Z(x) = 1 - \sum\limits_{i=1}^{n} \sum\limits_{p=1}^{r} \frac{A_i B_p \left(\frac{\beta_p}{\alpha_i}\right)}{x+\frac{\beta_p}{\alpha_i}}$. ☐

We also can conclude for this case of X/Y, the reliability function is $R_Z(t) = \sum\limits_{i=1}^{n} \sum\limits_{p=1}^{r} \frac{A_i B_p \left(\frac{\beta_p}{\alpha_i}\right)}{t+\frac{\beta_p}{\alpha_i}}$ and the hazard function is $h_Z(t) = \frac{\sum\limits_{i=1}^{n} \sum\limits_{p=1}^{r} \frac{A_i B_p \frac{\beta_p}{\alpha_i}}{\left(t+\frac{\beta_p}{\alpha_i}\right)^2}}{\sum\limits_{i=1}^{n} \sum\limits_{p=1}^{r} \frac{A_i B_p \left(\frac{\beta_p}{\alpha_i}\right)}{t+\frac{\beta_p}{\alpha_i}}}$.

The second particular case is when the Hypoexponential distribution has identical stages-the Erlang distribution. This case is when $\overrightarrow{\alpha} = \alpha$, $\overrightarrow{k} = k$ and $n = 1$ and we write $X \sim Hypoexp\,(\alpha, k) = Erl(k, \alpha)$, see [19]. Moreover, in [19], they showed that $A_{1j} = 0$ for $1 \leq j \leq k-1$ and $A_{1k} = 1$.

Corollary 4. *Let X and Y be two independent random variables with $X \sim Erl(k, \alpha)$ and $Y \sim Erl(l, \beta)$. Then $X/Y \sim GF(k, l, \frac{\beta}{\alpha})$.*

Proof. Theorem 1 gives the PDF of the general case of the Hypoexponential. Taking the particular case, the Erlang distribution, we get $A_{1j} = B_{1q} = 0$ for $1 \leq j \leq k-1$, $1 \leq q \leq l-1$ and $A_{1k} = 1$, $B_{1l} = 1$, $\overrightarrow{\alpha} = \alpha$, $\overrightarrow{\beta} = \beta$. Then the only terms of the summation in Eq (3.2) are when $j = k$ and $q = l$ having the coefficients $A_{1k}B_{1q} = 1$. Thus, the PDF of Z from Eq (3.2) can be written $f_Z(t) = f_{W_{k,l,i,p}}(t)$. We get $W_{k,l,i,p} \sim GF(k, l, \frac{\beta}{\alpha})$ and from Eq (2.5) the PDF is $f_Z(t) = \frac{\left(\frac{\beta}{\alpha}\right)^l t^{k-1}}{B(k,l)\left(t+\frac{\beta}{\alpha}\right)^{l+k}}$. ☐

The most particular case of the Hypoexponential distribution is the Exponential distribution, $X \sim Exp(\alpha)$. This particular case is $X \sim Hypoexp(\alpha, 1) = Hypoexp(\alpha) = Erl(1, \alpha)$. Thus, we obtain that when $X \sim Exp(\alpha)$ and $Y \sim Exp(\beta)$ are independent, then X/Y is log-logistic distribution with both parameters $\frac{\alpha}{\beta}$ and 1, having the PDF $f_{X/Y}(t) = \frac{\frac{\beta}{\alpha}}{\left(t+\frac{\beta}{\alpha}\right)^2}$.

3.3 Application

Next, we give an example to illustrate our work. Let X and Y be two independent Hypoexponential random variable with $X \sim Hypoexp(\overrightarrow{\alpha}, \overrightarrow{k})$, $\overrightarrow{\alpha} = (2, 3, 5)$, $\overrightarrow{k} = (2, 3, 1)$ and $Y \sim Hypoexp(\overrightarrow{\beta}, \overrightarrow{l})$, $\overrightarrow{\beta} = (2, 3, 4, 6)$, $\overrightarrow{l} = (1, 3, 4, 2)$. First, we find the coefficients of the linear combinations for the two random variables X and Y, by using Proposition 1. For X, the parameters are $\overrightarrow{\alpha}$ and $\overrightarrow{k}$. Thus the coefficients for X from Proposition 1 are $A = \{\{A_{11} = -300, A_{12} = 45\}, \{A_{21} = 405/2, A_{22} = 45, A_{23} = 10\}, \{A_{31} = -3/2\}\}$. Also from Proposition 1 the coefficients of Y with parameters $\overrightarrow{\beta}$ and $\overrightarrow{l}$ are $B = \{\{B_{11} = 972\}, \{B_{21} = -172032, B_{22} = 22528, B_{23} = -2048\}, \{B_{31} = 128304, B_{32} = 19440, B_{33} = 2430, B_{34} = 243\}, \{B_{41} = 156, B_{42} = 8\}\}$. Now the PDF of X/Y is given in Theorem 1 as $f_{X/Y}(t) = \sum_{i=1}^{3} \sum_{j=1}^{k_i} \sum_{p=1}^{4} \sum_{q=1}^{l_p} A_{ij}B_{pq}f_{W_{j,q,i,p}}(t)$, where $W_{j,q,i,p} \sim GF(j, q, \frac{\beta_p}{\alpha_i})$ represented in Fig.1.

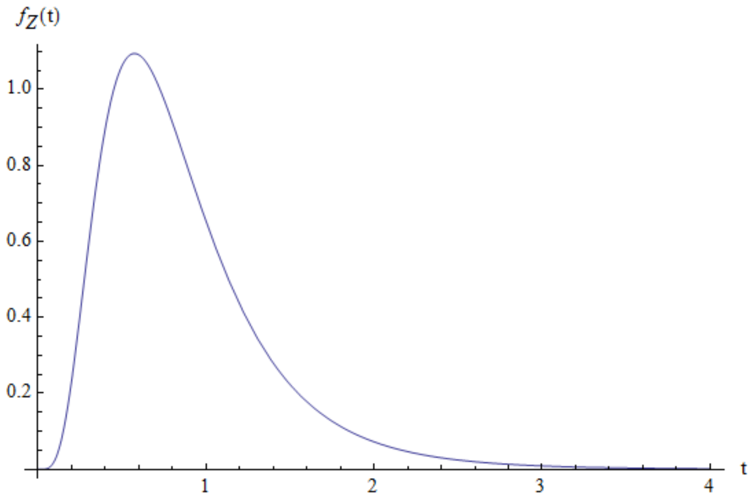

Figure 1: PDF of X/Y

4 Conclusion

The exact expressions of the PDF, CDF, moment generating function, reliability function and hazard function of the ratio of two independent general case of the Hypoexponential distribution are given. The expressions are given as a linear combination of the Generalized-F distribution. The expressions are applied for the particular case of the Hypoexponential distribution when the stages are different. Also the results are used to verify the other two particular cases (1) the Erlang distribution (2) the Exponential distribution.

Competing Interests

The authors declare that no competing interests exist.

References

[1] Ali MM, Pal M, Woo J. On the Ratio of Inverted Gamma Variates. Austrian Journal of Statistic. 2007;36(2):153-159.

[2] Idrizi L. On The Product and Ratio of Pareto and Kumaraswamy Random Variables. Mathematical Theory and Modeling. 2014;4(3):136-146.

[3] Shakil M, Golam Kibria BM. Exact Distribution of the Ratio of Gamma and Rayleigh Random Variables. Pakistan Journal of Statistics and Operation Research. 2006;2(2):87-98.

[4] Nadarajah S, Choi D. Arnold and Strauss's bivariate exponential distribution products and ratios. New Zealand Journal of Mathematics. 2006;35:189–199.

[5] Park S. On the Distribution Functions of Ratios Involving Gaussian Random Variables. ETRI Journal. 2010;32(6).

[6] Nadarajah S, Kotz S. On the product and ratio of t and Bessel random variables. Bulletin of the Institute of Mathematics Academia Sinica (New Series). 2007;2(1):55-66.

[7] Pham-Gia T, Turkkan N. Operations on the Generalized F-Variables, and Applications. A Journal of Theoretical and Applied Statistics. 2002;36(3):195-209. doi:10.1080/02331880212855.

[8] Marsaglia G. Ratios of Normal Variables and Ratios of Sums of Uniform Variables. Journal of the American Statistical Association. 1965;60(309):193-204.

[9] Korhonen PJ, Narula SC. The Probability Distribution of the Ratio of the Absolute Values of Two Normal Variables. Journal of Statistical Computation and Simulation. 1989;33(3):173-182. doi:10.1080/00949658908811195.

[10] Press SJ. The t-ratio distribution. Journal of the American Statistical Association. 1969;64(325):242-252.

[11] Basu AP, Lochner RH. On the distribution of the ratio of two random variables having generalized life distributions. Technometrics. 1971;13(2):281-287.

[12] Hawkins DL, Han CP. Bivariate distributions of some ratios of independent noncentral chi-square random variables. Communications in Statistics - Theory and Methods. 1986;15(1):261-277. doi:10.1080/03610928608829120.

[13] Provost SB. On the distribution of the ratio of powers of sums of gamma random variables. Pakistan Journal Statistic. 1989;5(2):157-174.

[14] Pham-Gia T. Distributions of the ratios of independent beta variables and applications. Communications in Statistics - Theory and Methods. 2000;29(12):2693-2715. doi:10.1080/03610920008832632.

[15] Nadarajaha S, Gupta AK. On the ratio of logistic random variables," Computational Statistics & Data Analysis. 2006;50:1206-1219.

[16] Nadarajaha S, Kotz S. On the Ratio of Frèchet Random Variables. Quality & Quantity. 2006;40:861-868.

[17] Nadarajah S. The linear combination, product and ratio of Laplace random variables. Statistics. A Journal of Theoretical and Applied Statistics. 2007;41(6):535-545.

[18] Coelho CA, Mexia JT. On the Distribution of the Product and Ratio of Independent Generalized Gamma-Ratio. Sankhyā: The Indian Journal of Statistics. 2007;69(2):221-255.

[19] Smaili K, Kadri T, Kadry S. A Modified-Form Expressions For The Hypoexponential Distribution. British Journal of Mathematics & Computer Science. 2014;4(3):322-332.

[20] Smaili K, Kadri T, Kadry S. Hypoexponential Distribution with Different Parameters. Applied Mathematics. 2013;4(4):624-631. doi: 10.4236/am.2013.44087.

[21] Pham-Gia T, Turkkan N. Distributions of Ratios: From Random Variables to Random Matrices. Open Journal of Statistics. 2011;1(2):93-104. doi: 10.4236/ojs.2011.12011.

[22] Pham-Gia T, Duong QP. The generalized beta and F-distributions in statistical modelling. Mathematical and Computer Modelling. 1989;12(12):1613-1625.

[23] Johnson N, Kotz S, Balakhrishnanand N. Continuous Univariate Distributions. John Wiley and Sons, New York, Vol. 2, 2nd ed, 1995.

[24] Press W, Teukolsky S, Vetterling W, Flannery B, Metcalf M. Numerical Recipes in C: The Art of Scientific Computing. Cambridge University Press, 3rd ed, 2007.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=584&id=6&aid=5235