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Approximation of Higher-order Singular Initial and Boundary Value Problems by Iterative Decomposition and Bernstein Polynomial Methods

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Abstract

In very recent time, various works have focused on the analysis of Singular Boundary Value Problems, with many techniques developed or used to deal with major questions relating to Singular Initial and Boundary Value Problems and their solutions. The main questions relate to existence and uniqueness of solution, the numerical approximation of solutions and convergence of solutions.

In this work, we focus on the last two questions for some classes of Singular Initial and Boundary Value Problems. We developed two approximation methods namely Iterative Decomposition and Bernstein Polynomial Methods and applied them to tackle the last two questions raised in this work.

Some numerical examples of second, third and fourth orders problems are considered to illustrate the efficiency and accuracy of the methods.

Keywords: Singular; iterative decomposition and Bernstein polynomial methods; approximation.

1 Introduction

Many models of physical phenomena in Physics, Engineering and other areas of Science are mathematically expressed as Singular two-point boundary and initial value problems associated with nonlinear second order ordinary differential equations of the type:

$$y'' = F(x, y, y'), \qquad a < x < b \tag{1.1}$$

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Singular initial and boundary value problems have received a lot of attention since the middle of the 20th century. Some analysis have been provided in [1,2,3] and also the authors discussed the existence and uniqueness of solutions (See [4]). The existence and uniqueness of solution were proved by [4]) using an iterative technique to solve singular boundary value problem.

Here, we consider a second order nonlinear ordinary differential equation of the form:

$$y''(x) + \phi(x)y^n = 0, \qquad 0 < x < 1 \tag{1.2}$$

$$y(0) = y(1) = 0 \tag{1.3}$$

Where $n > 0$ and $y(x)$ is positive and continuous for $0 < x < 1$. In Adomian (1998), Chawla and Katti (1982), a shooting method algorithm was constructed to solve equations (1.2) together with boundary conditions (1.3). The method was however found to be unstable.

A particularly popular form of (1.2) is the Lane-Emden-Fowler equation

$$y''(x) = ax^\sigma y^n(x), \qquad 0 < x < x_0 \tag{1.4}$$

where $\sigma,\ n < 0,\ a < 0$ and $x_0 > 0$ are real. It should be noted that the Emden-Fowler equation (1.4) arises in many models in Astrophysics, Nuclear Physics and Chemical reaction (See [5]).

2 Basic Definitions

The following definitions are essential to this work.

Definition 1.1

A matrix A is called analytic (or holomorphic) if every entry of it is an analytic function.

Definition 1.2

A point $x_0 \in (a, b)$ is an ordinary point of (1.1) if F is analytic at point $x = x_0$ otherwise, it is a singular point of (1.1). The simplest kind of singularities are isolated singularities, also called Pole-type singularities, A linear homogenous equation can be written in the form

$$y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0 \tag{2.1}$$

Definition 1.3

The differential equation (1.5) is said to have a regular point at $x = 0$ and is written in the form

$$x^2y''(x) + xb_1(x)y'(x) + b_2(x)y(x) = 0 \tag{2.2}$$

where b_1 and b_2 are analytic functions at $x = x$. A singular point that is not regular is called an irregular point.

Singular ordinary differential equations arise in the field of gas dynamics, Newtonian fluid, Mechanics, Fluid dynamics, Elasticity,

Reaction diffusion processes, Chemical kinetics, and other branches of applied Mathematics (See [4,6,1]), In particular, problem (1.1) arises in Physiological studies, for examples, the study of steady state oxygen diffusion in a spherical cell [3]. Recently, there has been much interest in the study of Singular two-point BVP's of types (1.2)-(1.3) (see for example [3,5,7] and many of the references therein).

The main difficulty in (1.1) is that the singularity behaviour occurs at $x = 0$.

A great number of methods have been applied to solve the singular boundary value problems. In [8], a standard three-point finite difference scheme was considered within uniform mesh for the solution. In [9], the finite difference methods were used to obtain numerical solutions. In [4] a numerical method based on Green's function was used to obtain a numerical solution of the same problem. A method, in which a modified decomposition method is applied with the cubic B-spline collocation technique is applied in [2] to obtain approximate solution with great accuracy.

Recently, the Adomian Decomposition Method (ADM) was applied to solve singular ODE (see [10,11]) applied a new modified decomposition method. [8] applied a fourth-order mesh while [12] used an extended Adomian Decomposition Method and these methods were modified further and applied by [7] to solve singular ODE.

The aforementioned numerical methods have many advantages, but a huge amount of computational work is required to obtain accurate numerical solutions, especially for nonlinear problems. This reason calls for the development of the Iterative Decomposition Method (IDM) which does not require construction of any form of polynomial. The constructed Bernstein polynomial is equally a simple method, which gives very accurate solutions. These two developed methods are applied in this work.

3 Materials and Methods

3.1 Iterative Decomposition Method

Consider the general functional equation

$$y = N(y) + f \tag{3.1.1}$$

where N is a nonlinear operator and f is a known function.

We need a solution y of (3.1.1) which has a series form

$$y = \sum_{n=0}^{\infty} y_n \tag{3.1.2}$$

The nonlinear operator N is decomposed as

$$N\left(\sum_{n=0}^{\infty} y_n\right) = N(y_0) + \sum_{n=0}^{\infty}\left\{N\left(\sum_{j=0}^{\infty} y_j\right) - N\left(\sum_{j=0}^{n-1} y_j\right)\right\} \tag{3.1.3}$$

From (3.1.2) and (3.1.3), (3.1.1) is equivalent to

$$\sum_{n=0}^{\infty} y_n = f + N(y_0) + \sum_{n=0}^{\infty}\left\{N\left(\sum_{j=0}^{\infty} y_j\right) - N\left(\sum_{j=0}^{n-1} y_j\right)\right\} \tag{3.1.4}$$

We then define the recurrence relation

$$\begin{aligned} y_0 &= f \\ y_1 &= N(y_0) \\ y_2 &= N(y_0 + y_1) - N(y_0) \\ \vdots &= \vdots \\ y_n &= N(y_0 + y_1 + \cdots + y_{n-1}) - N(y_0 + y_1 + \cdots + y_{n-2}) \end{aligned} \tag{3.1.5}$$

Thus,

$$y = f + \sum_{n=0}^{\infty} y_n \tag{3.1.6}$$

3.2 Discussion of IDM on General Second Order Singular Ordinary Differential Equation

For the purpose of this paper, we write the general second order singular ordinary differential equation considered as:

$$y'' + \frac{2n}{x} y' + \frac{n(n-1)}{x^2} y + f(x,y) = g(x), \;\; n > 0 \tag{3.2.1}$$

With the initial conditions given as:

$$y(0) = A, \qquad y'(0) = B \tag{3.2.2}$$

The problem is then written in an operator form as:

$$L(y) \; = \; g(x) \; - \; f(x,y)$$

where,

$$L = x^{-n} \frac{d^2}{dx^2} (x^n y)$$

For example, if $n = 1$, we have

$$y'' + \frac{2}{x} y' + f(x,y) = g(x)$$

If $n = 2$

$$y'' + \frac{4}{x} y' + \frac{2}{x^2} y + f(x,y) = g(x)$$

If $n = 3$

$$y'' + \frac{6}{x} y' + \frac{6}{x^2} y + f(x,y) = g(x)$$

and so on. We proposed the differential operator

$$L = x^{-n} \frac{d^2}{dx^2} (x^n y) \tag{3.2.3}$$

So, (3.2.1) is written as

$$L(y) = g(x) - f(x,y)$$

We assumed that the operator L is invertible and has an inverse L^{-1}, which is a two-fold integral operator

$$L^{-1}(\cdot) = x^{-n}\int_0^x \int_0^x x^n(\cdot)\,dx\,dx \tag{3.2.4}$$

Thus, (3.2.1) becomes

$$L^{-1}\left(y'' + \frac{2n}{x}y' + \frac{n(n-1)}{x^2}y\right) = x^{-n}\int_0^x \int_0^x x^n\left(y'' + \frac{2n}{x}y' + \frac{n(n-1)}{x^2}y\right)dx\,dx \tag{3.2.5}$$

Then we have,

$$y(x) = A + L^{-1}g(x) - L^{-1}f(x,y)$$

From (3.1.3), (3.1.4) – (3.1.6), we have

$$\begin{aligned} &y_0 = A + L^{-1}\{g(x)\} \\ &\vdots \\ &y_{k+1} = -\{L^{-1}(y_0 + y_1 + \cdots + y_n) - L^{-1}(y_0 + y_1 + \cdots + y_{n-1})\} \end{aligned} \tag{3.2.6}$$

3.3 Discussion of IDM on Higher-Order Singular IVP/BVP

Consider the singular BVP of $(n+1)$-th order ODE of the form

$$y^{(n+1)}(x) + \frac{m}{x}y^{(n)} + Ny = g(x) \tag{3.3.1}$$

where m is a constant and $n \geq 1$ together with the conditions

$$\begin{aligned} &y(0) = a_0, \quad y'(0) = a_1, \quad \cdots, \quad y^{(r-1)}(0) = a_{r-1} \\ &y(b) = c_0, \quad y'(b) = c_1, \quad \cdots, \quad y^{(n-r)}(b) = c_{n-r} \end{aligned} \tag{3.3.2}$$

where N is a nonlinear differential operator of order less than n, $g(x)$ is a given function, $a_0, a_1, a_2, \cdots a_{r-1}, c_0, c_1, \cdots, c_{n-r}, b$ are given constants, where $m \leq r \leq n, r \geq 1$

We first write (3.3.1) in the form

$$x^{-2}\frac{d^{n-1}}{dx^{n-1}}[x^2y'' + (m-2n+2)xy' + (n-m)(n-1)(n-1)y] + Ny = g(x) \tag{3.3.3}$$

or equivalently,

$$x^{-2}\frac{d^{n-1}}{dx^{n-1}}\left[x^{2n-m}\frac{d}{dx}\left(x^{m-2n+2}\frac{dy}{dx}\right)\right] + (n-m)(n-1)x^{-2}\frac{d^{n-1}y}{dx^{n-1}} + Ny = g(x) \tag{3.3.4}$$

which is written in the form

$$L_2L_1y = g(x) + (m-n)(n-1)L_2y - Ny \tag{3.3.5}$$

We employ the first two derivatives

$$L_1 = x^{2n-m}\frac{d}{dx}\left(x^{m-2n+2}\frac{d}{dx}\right) \tag{3.3.6}$$

and,

$$L_2 = x^{-2}\frac{d^{n-1}y}{dx^{n-1}} \tag{3.3.7}$$

To overcome the singularity at $x = 0$, by (3.3.6) and (3.3.7), the inverse operators L_1^{-1} and L_2^{-1} are the integral operators defined by

$$L_1^{-1}(\cdot) = \int_0^x x^{2n-m-2}\int_0^x x^{m-2n}(\cdot)dx\,dx \tag{3.3.8}$$

and

$$L_2^{-1} = \underbrace{\int_0^x \cdots \int_0^x x^2(\cdot)dxdx\cdots dx}_{(n-1)-times} \tag{3.3.9}$$

Applying L_2^{-1} on (3.3.5), we have

$$L_1 y = \psi_1(x) + L_2^{-1}g(x) - L_2^{-1}Ny \tag{3.3.10}$$

Here, $\psi_i(x) = (m-n)(n-1)y$

Such that

$$L_2\psi_1(x) = 0 \tag{3.3.11}$$

By applying L_1^{-1} on (3.3.10), we have

$$y(x) = L_1^{-1}\psi_1(x) + L_1^{-1}L_2^{-1}g(x) - L_1^{-1}L_2^{-1}Ny \tag{3.3.12}$$

Substituting the boundary and initial conditions into (3.3.12), we have y_0. We then proceed as for the second order singular IVP to apply the IDM.

3.4 Numerical Examples

3.4.1 Iterative decomposition method

Example 1:

Consider

$$y'' + \frac{2}{x}y' + y^3 = 6 + x^6 \tag{3.4.1}$$

$$y(0) = y'(0) = 0 \tag{3.4.2}$$

The exact solution is $y(x) = x^2$

We rewrite (3.4.1) as

$$Ly = 6 + x^6 - y^3 \tag{3.4.3}$$

where

$$L = x^{-1}\frac{d^2}{dx^2}(xy) \tag{3.4.4}$$

Since L is a linear operator, it is invertible, then the inverse L^{-1} is given as

$$L^{-1}(\cdot) = x^{-1}\int_0^x\int_0^x x(\cdot)dx\,dx \tag{3.4.5}$$

Operating operator on both sides of (3.4.1), we have

$$\begin{aligned}
y(x) &= L^{-1}\{6 + x^6\} - L^{-1}\{y^3\} \\
y(x) - y(0) - xy'(0) &= L^{-1}\{6 + x^6\} - L^{-1}\{y^3\} \\
y(x) &= x^{-1}\int_0^x\int_0^x x(6 + x^6)dx\,dx - L^{-1}\{y^3\} \\
&= x^{-1}\left(\frac{6x^3}{6} + \frac{x^9}{8.9}\right) - L^{-1}\{y^3\} \\
&= x^2 + \frac{x^8}{72} - L^{-1}\{y^3\}
\end{aligned}$$

Taking $y_0 = x^2 + \frac{x^8}{72}$,

Then, $y_1 = -L^{-1}\{y_0^3\}$

$$\begin{aligned}
y_2 &= [L^{-1}\{(y_0 + y_1)^3\} - L^{-1}\{y_0\}] \\
&= x^6 + \frac{x^{12}}{36} + \frac{x^{18}}{5184} + \frac{x^{12}}{72} + \frac{x^{18}}{2592} + \frac{x^{24}}{373248} \\
&= x^6 + \frac{x^{12}}{24} + \frac{x^{18}}{1728} + \frac{x^{24}}{373248} \\
&= -x^{-1}\left(\frac{x^9}{8\cdot 9} + \frac{x^{15}}{24\cdot 14\cdot 15} + \frac{x^{21}}{1728\cdot 20\cdot 21} + \frac{x^{27}}{373248\cdot 26\cdot 27}\right) \\
&= -x^{-1}\left(\frac{x^9}{72} + \frac{x^{15}}{5040} + \frac{x^{21}}{725760} + \frac{x^{27}}{262020096}\right)
\end{aligned}$$

$$y_1 = -L^{-1}\left\{\left(x^2 + \frac{x^8}{72}\right)^3\right\}$$

$$y_1 = -\frac{x^8}{72} - \frac{x^{14}}{5040} - \frac{x^{20}}{725760} - \frac{x^{26}}{262020096}$$

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} y_n \\
&= y_0 + y_1 + y_2 + \cdots
\end{aligned}$$

Thus, $y(x)$ is approximated as

$$y_0 + y_1 = x^2 + \frac{x^8}{72} - \frac{x^8}{72} - \frac{x^{14}}{5040} - \frac{x^{20}}{725760} - \frac{x^{26}}{262020096}$$
$$= x^2 - \frac{x^{14}}{5040} - \frac{x^{20}}{725760} - \frac{x^{26}}{262020096} \quad (3.4.6)$$

$$y(x) = x^2 - \frac{x^{14}}{5040} - \frac{x^{20}}{725760} - \frac{x^{26}}{262020096} \quad (3.4.7)$$

Which is very close to the exact solution, since the last three terms of the approximating polynomial makes very little contribution.

Example 2:

Consider the linear singular IVP

$$y'' + \frac{2}{x}y' + y = 6 + 12x + x^2 + x^3 \quad (3.4.8)$$

$$y(0) = y'(0) = 0 \quad (3.4.9)$$

The exact solution is $y(x) = x^2 + x^3$.

We write (3.4.8) in operator form as

$$Ly = 6 + 12x + x^2 + x^3 - y \quad (3.4.10)$$

Where

$$L = x^{-1}\frac{d^2}{dx^2}(xy) \quad (3.4.11)$$

Applying the inverse operator to both sides of (3.4.10), we have

$$y(x) = L^{-1}\{6 + 12x + x^2 + x^3\} - L^{-1}\{y\}$$
$$= x^{-1}\int_0^x\int_0^x x(6 + 12x + x^2 + x^3)dx\,dx - L^{-1}\{y\}$$
$$= x^{-1}\left(\frac{6x^3}{2\cdot 3} + \frac{12x^4}{3\cdot 4} + \frac{x^5}{4\cdot 5} + \frac{x^6}{5\cdot 6}\right) - L^{-1}\{y\} \quad (3.4.12)$$
$$= x^2 + x^3 + \frac{x^4}{20} + \frac{x^5}{30} - L^{-1}\{y\}$$

Taking

$$y_0 = x^2 + x^3 + \frac{x^4}{20} + \frac{x^5}{30} \quad (3.4.13)$$

and,

$$y_1 = -L^{-1\{y_0\}} \quad (3.4.14)$$

$$= \left\{x^{-1}\int_0^x\int_0^x x\left(x^2+x^3+\frac{x^4}{20}+\frac{x^5}{30}\right)dx\,dx\right\}$$
$$= -\left\{x^{-1}\left(\frac{x^5}{4\cdot 5}+\frac{x^6}{5\cdot 6}+\frac{x^7}{20\cdot 6\cdot 7}+\frac{x^8}{30\cdot 7\cdot 8}\right)\right\}$$
$$= -\frac{x^4}{20}-\frac{x^5}{30}-\frac{x^6}{840}-\frac{x^7}{1680}$$

$$\begin{aligned} y_2 &= -[L^{-1}\{(y_0+y_1)\}-L^{-1}\{y_0\}] \\ &= \left[L^{-1}\left\{x^2+x^3-\frac{x^6}{840}-\frac{x^7}{1680}\right\}-L^{-1}\{y_0\}\right] \\ &= -\left[\left\{x^{-1}\int_0^x\int_0^x x\left(x^2+x^3-\frac{x^6}{840}-\frac{x^7}{1680}\right)dx\,dx\right\}-L^{-1}\{y_0\}\right] \\ &= -\left[x^{-1}\left(\frac{x^5}{4\cdot 5}+\frac{x^6}{5\cdot 6}-\frac{x^9}{840\cdot 8\cdot 9}-\frac{x^{10}}{1680\cdot 9\cdot 10}\right)-L^{-1}\{y_0\}\right] \\ &= -\left[\frac{x^4}{20}+\frac{x^5}{30}-\frac{x^8}{60480}-\frac{x^9}{151200}-\left(\frac{x^4}{20}+\frac{x^5}{30}+\frac{x^6}{840}+\frac{x^7}{1680}\right)\right] \\ &= -\left[-\frac{x^6}{840}-\frac{x^7}{1680}-\frac{x^8}{60480}-\frac{x^9}{151200}\right] \end{aligned} \tag{3.4.15}$$

$$y_2 = \frac{x^6}{840}+\frac{x^7}{1680}+\frac{x^8}{60480}+\frac{x^9}{151200}$$

$$\begin{aligned} y_3 &= -[L^{-1}\{(y_0+y_1+y_2)\}-L^{-1}\{y_0+y_1\}] \\ &= -\left[\frac{x^4}{20}+\frac{x^5}{30}+\frac{x^{10}}{6652800}+\frac{x^{11}}{19958400}-\left(\frac{x^4}{20}+\frac{x^5}{30}-\frac{x^8}{60480}-\frac{x^9}{15120}\right)\right] \\ &= -\frac{x^5}{60480}-\frac{x^9}{151200}-\frac{x^{10}}{6652800}-\frac{x^{11}}{19958400} \end{aligned} \tag{3.4.16}$$

Then, we approximate $y(x)$ as

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} y_n \\ &= x^2+x^3-\frac{x^{10}}{6652800}-\frac{x^{11}}{19958400} \end{aligned} \tag{3.4.17}$$

Remark 1: Here, we tabulated the exact solutions with the approximate solutions in $[0,1]$ in order to ascertain the accuracy of the proposed method. We have defined error as $|y(x) - y_n(x)|$ in this work.

Table 1. Table of values for example 2

x	Exact	Approx. solution	Error
0.0	0.000	0.0000000000	0.0000
0.1	0.011	0.0100000000	0.0000
0.2	0.048	0.0480000000	0.0000
0.3	0.117	0.1170000000	0.0000
0.4	0.224	0.2240000000	0.0000
0.5	0.375	0.3749999998	2.00E-10
0.6	0.576	0.5759996880	3.12E-7
0.7	0.833	0.8330000210	2.10E-8
0.8	1.152	1.1519999991	1.00E-9
0.9	1.533	1.5330000037	3.00E-9
1.0	2.000	2.0000000001	1.00E-10

From the table above, it is obvious that our solutions are quite close to the exact solutions.

Example 3:

Consider the linear singular IVP

$$y''' + \frac{\cos x}{\sin x} y'' = \sin x \cos x \tag{3.4.18}$$

$$y(0) = 1, \qquad y'(0) = -2, \qquad y''(0) = 0 \tag{3.4.19}$$

while the exact solution is

$$y(x) = 1 - 2x + \frac{x^2}{12} - \frac{\sin^2 x}{12}$$

Solution:

$$L = x^{-2} \frac{d}{dx}\left[x^2 \frac{d^2}{dx^2}\right] \tag{3.4.20}$$

Then equation (3.4.18) is written in operator form as

$$Ly = \sin x \cos x \tag{3.4.21}$$

Applying the inverse operator given as

$$\begin{aligned} L^{-1}(\cdot) &= \int_0^x \frac{(x-t)^{n-2}}{(n-2)!} e^{\int p(s)ds} (\cdot) ds\, dt \\ &= \int_0^x \frac{x-t}{\sin t} \int \sin s(\cdot)\, ds\, dt \end{aligned}$$

on both sides of equation (3.4.21), we have

$$\begin{aligned} y(x) &= y(0) + xy'(0) + x^2 y''(0) + L^{-1}(\sin x \cos x) \\ &= 1 - 2x - \frac{1}{2x} - \frac{1}{12}x^2 + \frac{1}{12}\cos^2 x \\ &= 1 - 2x - \frac{1}{12} + \frac{1}{12}x^2 + \frac{1}{12}(1 - \sin^2 x) \\ &= 1 - 2x - \frac{1}{2} + \frac{x^2}{12} - \frac{\sin^2 x}{12} \end{aligned}$$

which is the exact solution.

Example 4:

Consider the nonlinear BVP

$$y''' + \frac{3}{x} y'' - y^3 = g(x) \tag{3.4.22}$$

$$y(0) = 0, \qquad y'(0) = 0, \qquad y(1) = e \tag{3.4.23}$$

where

$$g(x) = 24e^x + 36xe^x + 12x^2e^x + x^3e^x - x^9e^{3x}$$

The exact solution of the problem is

$$y(x) = x^3e^x \tag{3.4.24}$$

Solution:

Taking

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

in $g(x)$, we have

$$g(x) = \frac{18}{x} + 48 + 50x + 30x^2 + \frac{49}{4}x^3 + \frac{71}{15}x^4 + \frac{29}{10}x^5 + \frac{33}{28}x^6 - \frac{6599}{4032}x^7 - \frac{9649}{2520}x^8 - \frac{1279991}{302400}x^9 \tag{3.4.25}$$

Then, applying the inverse operator as in the previous example, we have

$$\begin{aligned} y_0 &= 0.8888889x^3 + 0.9375000x^4 + 0.4800000x^5 + 0.1620370x^6 + 0.0408163x^7 \\ &\quad + 0.0082031x^8 + 0.0013717x^9 + 0.0001964x^{10} + 0.0000246x^{11} + \cdots \\ y_1 &= 0.0987654x^3 + 0.0585938x^4 + 0.00192000x^5 + 0.0045010x^6 + 0.0008329x^7 \\ &\quad + 0.0001282x^8 + 0.0000169x^9 + 0.0000019x^{10} + \cdots \\ y_2 &= 0.0109739x^3 + 0.0036621x^4 + 0.0007680x^5 + 0.0001250x^6 + 0.0000169x^7 \\ &\quad + 0.0000020x^8 + \cdots \end{aligned}$$

Then, the solution $y(x)$ is approximated as

$$\begin{aligned} y(x) &= \sum_{n=0}^{3} y_n(x) \\ &= 0.9998476x^3 + 0.9999847x^4 + 0.4999987x^5 + 0.1666666x^6 + 0.04166667x^7 \\ &\quad + 0.0083333x^8 + 0.0013888x^9 + 0.0001984x^{10} + \cdots \end{aligned}$$

Remark 2: Here, we tabulated the exact solutions with the approximate solutions in [0, 1] in order to ascertain the accuracy of the proposed method.

Table 2. Table of values for example 4

x	Exact	Approx. solution	Error
0.0	0.000000000000	0.000000000000	0.00000000
0.1	0.001105170918	0.001105016975	1.53943E-7
0.2	0.009771222064	0.009769977963	1.24410E-6
0.3	0.036446187820	0.036441945820	4.24200E-6
0.4	0.095476780670	0.095466620650	1.01600E-5
0.5	0.206090158900	0.206070098000	2.00061E-5
0.6	0.393537766080	0.393542557500	3.51033E-5
0.7	0.690717178500	0.690660468600	5.67099E-5
0.8	1.139476955000	1.139389862000	8.70930E-5
0.9	1.793050668000	1.792920038000	1.30630E-4
1.0	2.718281828000	2.717084770000	1.97050E-4

Example 5:

Consider the nonlinear BVP

$$y^{(iv)}(x)+\frac{3}{x}y'''(x)+y^2-y^3=\frac{18}{x}e^x+30e^x+11xe^x+x^2e^x+x^4e^{2x}-x^6e^{3x} \quad (3.4.26)$$

subject to the boundary conditions

$$y(0)=0, \qquad y(1)=e, \qquad y'(1)=3e \quad (3.4.27)$$

The exact solution of the problem is $y(x)=x^2e^x$.

Solution:

Expanding the right hand side of the equation (3.4.26) in a power series for easy handling, we have

$$\begin{aligned}y^{(iv)}(x)+\frac{3}{x}y'''(x)+y^2-y^3 &= \frac{18}{x}+48+50x+30x^2+\frac{49}{4}x^3+\frac{71}{15}x^4+\frac{29}{10}x^5+\frac{33}{28}x^6-\frac{6599}{4032}x^7-\frac{9649}{2520}x^8 \\ &\quad -\frac{1279991}{302400}x^9\end{aligned}$$

From the definition of the linear operators, we have

$$L_1=x^3\frac{d}{dx}\left(x^{-1}\frac{d}{dx}\right) \quad (3.4.28)$$

$$L_2=x^{-2}\frac{d^2}{dx^2} \quad (3.4.29)$$

So that

$$L_1^{-1}(\cdot)=\int_0^x x\int_0^x x^{-3}(\cdot)dx\,dx \quad (3.4.30)$$

$$L_2^{-1}(\cdot)=\int_0^x\int_0^x x^2(\cdot)dx\,dx \quad (3.4.31)$$

Putting (3.4.26) in operator form, we have

$$\begin{aligned}L_2L_1y &= \frac{18}{x}+48+50x+30x^2+\frac{49}{4}x^3+\frac{71}{15}x^4+\frac{29}{10}x^5+\frac{33}{28}x^6-\frac{6599}{4032}x^7-\frac{9649}{2520}x^8 \\ &\quad -\frac{1279991}{302400}x^9+y-y^2\end{aligned} \quad (3.4.32)$$

Applying the inverse operator equations (3.4.30) and (3.4.32) in that order, we have

$$\begin{aligned}y_0 &= 0.9993606x^2+x^3+0.5x^4+0.1666667x^5+0.0416667x^6+0.0083333x^7+0.0017609x^8 \\ &\quad +0.0006393x^9+0.0001637x^{10}-0.0001503x^{11}-0.0002417x^{12} \\ &\quad -0.0001897x^{13}\end{aligned}$$

$$y_1=-0.0007032x^2-0.0003715x^8-0.0004406x^9-0.0001391x^{10}+0.0001527x^{11}+\cdots$$

$$y_2 = \left(2.946962x10^{-6}\right)x^2 + \left(5.22889299x10^{-7}\right)x^8 + \left(3.100587x10^{-7}\right)x^9 - \left(1.949623x10^{-7}\right)x^{10}$$

$$y_3 = -\left(1.205504x10^{-8}\right)x^2 - \left(2.375252x10^{-9}\right)x^8 - \left(1.299369x10^{-9}\right)x^9 - \left(1.022952x10^{-9}\right)x^{10} + \left(1.66866x10^{-9}\right)x^{11}$$

Then $y(x)$ is approximated as

$$y(x) = 0.9986603x^2 + x^3 + 0.5x^4 + 0.1666667x^5 + 0.0416667x^6 + 0.008333x^7 - 0.0013899x^8 + 0.00019901x^9 + 0.000024x^{10} + \left(2.059130x10^{-6}\right)x^{11}$$

Remark 3: Here, we tabulated the exact solutions with the approximate solutions in [0, 1] in order to ascertain the accuracy of the proposed method.

3.5 Bernstein Polynomial Method

In this section, we describe the Bernstein polynomial method generally. We define the Bernstein polynomial as

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \qquad i = 0,1,2,\cdots,m \tag{3.5.1}$$

where the binomial coefficient is

$$\binom{m}{i} = \frac{m!}{i(m-i)!} \tag{3.5.2}$$

There are $(m+1)$th degree of Bernstein polynomials.

For mathematical convenience, we usually set

$$B_{i.m} = 0, \text{ if } i < 0 \text{ or } i > m$$

In general, we represent any function $u(x)$ with the first $(m+1)$ Bernstein polynomial as

$$u(x) = \sum_{i=0}^{m} C_i B_{i,m}(x) = C^T \phi(x) \tag{3.5.3}$$

Where $C^T = [c_0, c_1, \cdots, c_m]$, $\phi(x) = \left[B_{0,m}(x), B_{i,m}(x), \cdots B_{m,n}(x)\right]$

The derivative of the vector $\phi(x)$ is expressed as

$$\frac{d}{dx}\phi(x) = D'\phi(x) \tag{3.5.4}$$

where D' is the $(m+1)$-square operational matrix of derivatives, defined as

$$\phi(x) = \begin{pmatrix} B_{0,m} \\ B_{i,m} \\ \vdots \\ B_{m,n} \end{pmatrix} = \begin{pmatrix} a_0 + a_1 x + \cdots + a_m x^m \\ b_0 + b_1 x + \cdots + b_m x^m \\ \vdots \\ z_0 + z_1 x + \cdots + z_m x^m \end{pmatrix}$$

$$= \begin{pmatrix} a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_m \\ \vdots & \vdots & \ddots & \vdots \\ z_0 & z_1 & \cdots & z_m \end{pmatrix} \begin{pmatrix} 1 \\ x \\ \vdots \\ x^m \end{pmatrix} \quad (3.5.5)$$

where $a_0, a_1, \cdots, a_n, b_0, b_1, \cdots, b_m$ and $z_0, z_1, \cdots, z_m$ are the binomial coefficients of the Bernstein polynomial.

Now,

$$\frac{d}{dx}\phi(x) = \begin{pmatrix} a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_m \\ \vdots & \vdots & \ddots & \vdots \\ z_0 & z_1 & \cdots & z_m \end{pmatrix} \frac{d}{dx} \begin{pmatrix} 1 \\ x \\ \vdots \\ x^m \end{pmatrix}$$

$$= \begin{pmatrix} a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_m \\ \vdots & \vdots & \ddots & \vdots \\ z_0 & z_1 & \cdots & z_m \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ mx^{m-1} \end{pmatrix}$$

$$= \begin{pmatrix} a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_m \\ \vdots & \vdots & \ddots & \vdots \\ z_0 & z_1 & \cdots & z_m \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & m & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ \vdots \\ x^m \end{pmatrix}$$

$$= \begin{pmatrix} a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_m \\ \vdots & \vdots & \ddots & \vdots \\ z_0 & z_1 & \cdots & z_m \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & m & 0 \end{pmatrix} \begin{pmatrix} a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_m \\ \vdots & \vdots & \ddots & \vdots \\ z_0 & z_1 & \cdots & z_m \end{pmatrix}^{-1} \phi(x)$$

$$= x \begin{pmatrix} a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_m \\ \vdots & \vdots & \ddots & \vdots \\ z_0 & z_1 & \cdots & z_m \end{pmatrix} \begin{pmatrix} 1 \\ x \\ \vdots \\ x^m \end{pmatrix} \quad (3.5.6)$$

$$= \begin{pmatrix} a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_m \\ \vdots & \vdots & \ddots & \vdots \\ z_0 & z_1 & \cdots & z_m \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & m & 0 \end{pmatrix} \begin{pmatrix} a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_m \\ \vdots & \vdots & \ddots & \vdots \\ z_0 & z_1 & \cdots & z_m \end{pmatrix}^{-1} \phi(x)$$

$$= D^{'}\phi(x) \quad (3.5.7)$$

This is generalized as

$$\frac{d^2\phi(x)}{dx^2} = (D^{'})^2\phi(x)$$
$$= (D^{'})(D^{'})(x)$$

and thus

$$\frac{d^n}{dx^n}\phi(x) = (D^{'})^n\phi(x)$$
$$= \underbrace{(D^{'})(D') \cdots (D^{'})\phi(x)}_{n-times}$$

Given a general singular two-point boundary value problem,

$$= \frac{1}{p(x)} u''(x) + \frac{1}{q(x)} u'(x) + \frac{1}{r(x)} \left(u(x)\right)^k = g(x), \qquad 0 < x \leq 1 \tag{3.5.8}$$

subject to the boundary conditions

$$u(0) = \alpha, \qquad u(1) = \beta \tag{3.5.9}$$

where p, q, r and g are continuous function on $[0,1]$. The parameters α, β are real constants, k is an integer. From the definitions of the derivatives of the function $u(x)$ in terms of Bernstein polynomial

$$u''(x) \approx C^T (D')^2 \phi(x) \tag{3.5.10}$$

$$u'(x) \approx C^T D' \phi(x) \tag{3.5.11}$$

and

$$\left(u(x)\right)^k = \left(C^T \phi(x)\right)^k$$

We define the residual of (3.5.8) as

$$R(x) = \frac{1}{p(x)} C^T (D')^2 \phi(x) + \frac{1}{q(x)} C^T D' \phi(x) + \frac{1}{r(x)} \left(C^T \phi(x)\right)^k - g(x) \tag{3.5.12}$$

To find the solution $u(x)$, we generate $(m-1)$ linear equation by applying

$$\int_0^1 R(x) B_{i,m}\, dx = 0, \qquad i = 0, 1, 2, \cdots, m-2 \tag{3.5.13}$$

By substituting the boundary value condition into (28) and (29), we have

$$u(0) = C^T \phi(0) = \alpha \tag{3.5.14}$$

$$u(1) = C^T \phi(1) = \beta \tag{3.5.15}$$

The residual equation with the boundary conditions generate $(m+1)$ of equations.

These equations are then solved for the unknown coefficients of the vector C^T. Consequently $u(x)$ is easily calculated.

3.6 Numerical Examples

Example 6:

Consider the linear two-point BVP

$$y'' + \frac{1}{x} y' + y(x) = 4 - 9x + x^2 + x^3 \tag{3.6.1}$$

with boundary conditions

$$y(0) = y'(1) = 0 \tag{3.6.2}$$

The exact solution is $y(x) = x^2 - x^3$.

Applying the technique of the Bernstein polynomial, we have for $m = 3$

$$y(x) = c_0 B_{0,3}(x) + c_1 B_{1,3}(x) + c_3 B_{3,3}(x) = C^T \phi(x) \tag{3.6.3}$$

Then,

$$D' = \begin{pmatrix} -3 & -1 & 0 & 0 \\ 3 & -1 & -2 & 0 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$(D')^2 = \begin{pmatrix} 6 & 4 & 2 & 0 \\ -12 & -6 & 0 & 6 \\ 6 & 0 & -6 & -12 \\ 0 & 0 & 4 & 6 \end{pmatrix} \tag{3.6.4}$$

Using (3.5.14), we have

$$-\frac{3}{56} + \frac{79}{280}c_0 + \frac{c_1}{56} + \frac{9c_2}{56} + \frac{43c_3}{280} = 0 \tag{3.6.5}$$

$$\frac{11}{280} + \frac{c_0}{56} - \frac{15c_1}{56} - \frac{33c_2}{280} + \frac{131c_3}{280} = 0 \tag{3.6.6}$$

Also, using (3.5.10) and (3.5.11), we have

$$c_0 = 0, \qquad c_3 = 0 \tag{3.6.7}$$

Thus, solving (3.6.5) and (3.6.6), we have

$$c_0 = 0, \qquad c_1 = 0, \qquad c_2 = \frac{1}{3}, \qquad c_3 = 0 \tag{3.6.8}$$

Then,

$$y(x) = \left(0,0,\frac{1}{3},0\right)\begin{pmatrix} 1 - 3x + 3x^2 - x^3 \\ 3x - 6x^2 + 3x^3 \\ 3x^2 - 3x^3 \\ x^3 \end{pmatrix} = (x^2 - x^3) \tag{3.6.9}$$

which is the exact solution.

Example 7:

Consider the singular boundary value problem

$$\left(1 - \frac{x}{2}\right) y''(x) + \frac{3}{2}\left(\frac{1}{x} - 1\right) y'(x) + \left(\frac{x}{2} - 1\right) y(x) = g(x), \qquad 0 < x \leq 1 \tag{3.6.10}$$

Subject to the boundary conditions

$$y(0) = 0, \qquad y'(1) = 0 \tag{3.6.11}$$

The function $g(x)$ is given as

$$g(x) = 5 - \frac{29x}{2} + \frac{13x^2}{2} + \frac{3x^3}{2} - \frac{x^4}{2}$$

The exact solution is $y(x) = x^2 - x^3$.

By the techniques of the Bernstein polynomial, we find

$$c_0 = 0, \qquad c_1 = 0, \qquad c_2 = \frac{1}{3}, \qquad c_3 = 0 \tag{3.6.12}$$

and

$$y(x) = \left(0,0,\frac{1}{3},0\right)\begin{pmatrix} 1 - 3x + 3x^2 - x^3 \\ 3x - 6x^2 + 3x^3 \\ 3x^2 - 3x^3 \\ x^3 \end{pmatrix} = x^2 - x^3$$

This is also the exact solution.

4 Conclusion

We have demonstrated the two proposed methods on seven numerical examples. Firstly, examples 1,2 are second-order singular ordinary differential equation with Non-Homogeneous and the methods performed credibly well when compared with exact solution.

Secondly, example 3 is second-order non-homogeneous with trigonometric functions. Many numerical methods have avoided this problem because of the trigonometric functions involved. With the proposed methods, we got the exact solution with ease.

Thirdly, examples 4 and 5 are third and fourth orders singular ordinary differential respectively. We applied our proposed methods to these problems and the results obtained are very close to the exact solution as this is evident in Tables 1 and 2 respectively above.

Fourthly, examples 6 and 7 are second order singular ordinary differential equations. We applied the Bernstein polynomial method to these two problems and we got exact solutions.

In conclusion, the two proposed methods are accurate, efficient and reliable to tackle the classes of problems discussed in this paper.

Competing Interests

Authors have declared that no competing interests exist.

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