Journal of Advances in Mathematics and Computer Science

25(5): 1-14, 2017; Article no.JAMCS.37398

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Full Observability Criteria for Linear Impulsive Control Systems with Application to Diabetes Type I Dynamics

P. S. Rivadeneira[1,2*]

[1] *Facultad de Minas, Universidad Nacional de Colombia, Grupo GITA, Cra. 80 # 65-223, Medellín, Colombia.*

[2] *INTEC-Facultad de Ingeniería Química (UNL-CONICET), Güemes 3450, 3000 Santa Fe, Argentina.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/JAMCS/2017/37398

Editor(s):

(1) M. Subhas Abel, Professor, Department of Mathematics, Gulbarga University, India.

(2) Mohamed Rabea Eid Said, Professor, Department of Science and Mathematics, Assiut University, Egypt.

(3) Tian-Xiao He, Professor, Department of Mathematics and Computer Science, Illinois Wesleyan University, USA.

Reviewers:

(1) Özlem Esen, Anadolu University, Turkey.

(2) Hazem Ibrahim Ali, University of Technology, Iraq.

(3) Kai-Ning Wu, Harbin Institute of Technology, China.

(4) Maria Isabel Garcia-Planas, Polytechnic University of Catalonia, Spain.

Complete Peer review History: http://www.sciencedomain.org/review-history/22177

Received: ***13th October 2017***

Accepted: ***21st November 2017***

Published: ***7th December 2017***

Original Research Article

Abstract

The main goal of this paper is to derive a full characterization of the observability of linear time-invariant impulsive systems. Two cases are studied. The first case considers continuous outputs. A suitable adapted Kalman criterion is shown to characterize properly the observability. In the second case, only discrete-time measurements of the outputs are available. A new rank condition based on the structure of the impulses is shown to characterize observability. Finally, these results are tested and illustrated both on academic examples and on the dynamical model of diabetic type I patients. The latter provides a nice case study for an impulsive system with discrete time measurements as the meals may be approximated by some impulse inputs and the glycemia measurement is usually done in real life at various times through the day.

Corresponding author: E-mail: psrivade@unal.edu.co

Keywords: Impulsive control systems; linear systems; observability; estimation of parameters; diabetes mellitus type I.

2010 Mathematics Subject Classification: 93C05, 93B07

1 Introduction

Impulsive systems are encountered in various areas as biology, health, robotics and others [1, 2, 3, 4, 5]. For instance, a diabetic type I will be shortly considered herein for which new specific mathematical tools are needed for analysis, observation, and control. The meals decide about the amount of glucose which is brought, say three times a day, and due to different time scales, it can be approximated by impulse signals. The glycemia measurement is made from blood samples taken at various times during the day.

Several mathematical models have been proposed to describe the dynamics of the glucose-insulin interaction for healthy and diabetic type I persons. One of the first models was published in 1961 by V. Bolie [6] and remains as the simplest one. More sophisticated models have been proposed, see for instance [7, 8, 9, 10, 11].

From a mathematical point of view, an impulsive control system is modeled either by continuous-time equations whose right-hand side includes some Dirac impulses, or by a system of ordinary differential equations and algebraic discrete equations. Its state trajectories are piecewise continuous, with discontinuities of the first kind at some isolated points. The basic mathematical tools for studying impulsive control systems (ICS) is the theory of impulsive differential equations [12]. The theory of linear ICS has been developed during this last decade through the investigation of fundamental properties such as stability, controllability, and observability (see [2, 13, 14, 15, 16, 17] and their references).

Observability in nonlinear ICS has been studied only by [18], while in linear ICS this property has been investigated by many researchers as [13, 15, 16, 19, 20, 21]. The definition used in these papers establishes that observability depends on measurements of the output on a finite-time interval $[0, t_f]$. When continuous output is considered, the most known result to characterize this property is still the Kalman observability matrix $\mathcal{O}$ [13, 21, 19], but with a very restrictive assumption over the class of impulsive system considered. Just diagonal matrices A_I are considered, where A_I defines a discontinuity of the form $x(\tau_k^+) = A_I x(\tau_k)$. In [22, 20], a criterion is derived assuming that the matrices A and A_I conmute, and finally, in [16], based on the product of matrices $e^{(A)} A_I^T \mathcal{O}^T$ an algebraic criterion is proposed. A different class of impulsive control systems is considered in [23], for which the states evolve in continuous form but the output is available for measurement at discrete times. Suitable criteria based on geometric properties of the invariant observable space and the observability Gramian were worked out for this case.

The results of this paper are: first, the notion of strong observability is introduced for continuous and discrete outputs. Kalman's criterion is a necessary condition for strong observability similar as the standard LTI case. Second, observability on a finite-time interval is analyzed. For the continuous output case, Kalman's criterion becomes just a sufficient condition. A new rank condition is derived considering just the time shifts and shown to be necessary for observability of ICS. This generalizes some results in the current literature. Also, the discrete output case is tackled, an equivalence between the observability Gramian and a new algebraic condition is established. Third, these criteria are tested in a model of diabetic type I patient (see [6]) adapted to the description of linear ICS. Based on the observation of the states, an estimation of the parameters of the model proposed was performed by using clinical data.

The remainder of the paper is organized as follows: the state of art of observability in linear ICS is given in Section 2. The theoretical framework and the main results for linear ICS are developed in Section 3. The results are illustrated by using an adaptation of Bolie's model of diabetic type I patients in Section 4. The last Section is devoted to conclusions and perspectives.

2 Preliminaries

A plant is an impulsive control system when there is a set of time instants $T = \{\tau_k\}$, $\tau_k \in \mathbb{R}$, $\tau_k < \tau_{k+1} < \infty$, and a set of inputs $U_k \in \mathbb{R}^n$, $k = 1, 2, \ldots$, such that the state $x \in \mathbb{R}^n$ at each τ_k is changed impulsively by $x(\tau_k^+) = f_I(x(\tau_k)) + U(k, x)$. Note that the impulsive instants are not necessarily equidistant, the control $U(k, x)$ yields a discontinuity of x at instant τ_k, the function $f_I(x)$ defines discontinuities of the first kind (or 'natural jumps') in the state variables, and the system is left-continuous, *i.e.* $x(\tau_k^-) = x(\tau_k)$.

The class of dynamic systems of interest basically consists of objects defined by a set of impulsive first-order differential equations of the form

$$\begin{cases} \dot{x}(t) &= Ax(t), \quad x(t_0^+) = x(t_0) = x_0, \quad t \neq \tau_k, \\ x(\tau_k^+) &= A_I x(\tau_k) + Bu(\tau_k), \qquad k \in \mathbb{N}, \\ y_c(t) &= C_c x(t) \qquad \textit{or} \\ y_d[k] &= C_d x(\tau_k) \qquad k \in \mathbb{N} \end{cases} \tag{2.1}$$

where the independent variable $t \in \mathbb{R}$ denotes time, the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, the variable $y_c \in \mathbb{R}^q$ is a continuous output, and y_d is a set of discrete measurements. The matrices A, A_I are defined in $\mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{q\times n}$. This system is known as the linear time-invariant ICS. Note that when $A_I = I$, there are not natural jumps in the state variables, only due to the control $u(\tau_k)$.

Let us denote the initial time as $t_0 = 0$, the final time as $t_f = \tau_{k+1}^- > t_0$, the set of time instants as $T = \{\tau_1, \cdots, \tau_k\}$, with δ_i being $\delta_i = \tau_{i+1} - \tau_i$, which verifies $\delta_0 = \tau_1 - t_0$, and $\delta_k = t_f - \tau_k$. Also, consider that $A_I \neq I$ and $B \neq 0$, then the state response for these kind of systems can be generated as follows:

- In $t = t_0$, any control is applied. So, in the interval $t_0 \leq t < \tau_1$, the state response is

$$x(t) = \Phi(t, t_0)x_0 = e^{A(t-t_0)}x_0. \tag{2.2}$$

- In $t = \tau_1$,

$$x(\tau_1^+) = A_I x(\tau_1) + Bu(\tau_1) = A_I e^{A\delta_0} x_0 + Bu(\tau_1). \tag{2.3}$$

- In the interval $t_0 \leq t < \tau_2$, with one impulse applied to Eq. (2.1), the state response is

$$\begin{aligned} x(t) &= e^{A(t-\tau_1)} A_I e^{A\delta_0} x_0 + e^{A(t-\tau_1)} Bu(\tau_1), & (2.4) \\ &= \Phi(t, t_0)x_0 + \Phi(t, \tau_1)Bu(\tau_1). & (2.5) \end{aligned}$$

- In $t = \tau_2$,

$$\begin{aligned} x(\tau_2^+) &= A_I x(\tau_2) + Bu(\tau_2), & (2.6) \\ &= A_I(e^{A\delta_1} A_I e^{A\delta_0} x_0 + e^{A\delta_1} Bu(\tau_1)) + Bu(\tau_2). \end{aligned}$$

- In the interval $\tau_0 \leq t < \tau_3$, with two impulses applied to Eq. (2.1), the state response is

$$\begin{aligned} x(t) &= e^{A(t-\tau_2)}A_I e^{A\delta_1}A_I e^{A\delta_0}x_0 + \\ &\quad e^{A(t-\tau_2)}e^{A\delta_1}Bu(\tau_1) + e^{A(t-\tau_2)}Bu(\tau_2), \\ &= \Phi(t,t_0)x_0 + \Phi(t,\tau_1)Bu(\tau_1) + \Phi(t,\tau_2)Bu(\tau_2). \end{aligned} \tag{2.7}$$

By repeating this procedure, the state transition matrix of Eq. (2.1) is deduced for the general interval $t_0 \leq t \leq t_f < \tau_{k+1}$, with k impulses applied to the system, and it is given by

$$\Phi(t,t_0) = e^{A\delta_k}A_I e^{A\delta_{k-1}} \cdots A_I e^{A\delta_1}A_I e^{A\delta_0}. \tag{2.8}$$

The state transition matrix is invertible for all $t \in [t_0, t_f]$ if only if the matrix A_I is invertible, and in this case, $\Phi(t_0,t) = \Phi^{-1}(t,t_0)$. The state response of system (2.1) on $[t_0, t]$ with k impulses applied to the system is

$$x(t) = \Phi(t,t_0)x_0 + \sum_{j=1}^{k}\Phi(t,\tau_j)Bu(\tau_j). \tag{2.9}$$

Note that if $B = 0$ and $A_I = I$, the state transition matrix for LTI systems is recovered, that is, $\Phi_c(t,t_0) = e^{A(t-t_0)}$ and the state response is just $x(t) = e^{A(t-t_0)}x_0$. Now, if $B \neq 0$ but $A_I = I$, the state response equation becomes

$$x(t) = e^{A(t-t_0)}\left(x_0 + \sum_{j=1}^{k}e^{-A\tau_j}Bu(\tau_j)\right), \tag{2.10}$$

which agrees with the results in [4].

3 Observability for Linear ICS

The notion of observability in standard systems concerns the possibility of recovering the state $x(t)$ from the knowledge of the measured output (being continuous or discrete), the input u, and, possibly, a finite number of their time derivatives.

Here, for linear ICS, this notion is reduced to finding the initial conditions using the knowledge of the output, its time derivatives, and/or its time-shifts.

3.1 Observability with continuous output

The case with continuous output will be treated, i.e $y_c(t) = C_c x(t)$, with C_c a matrix of $q \times n$ dimensions. In a similar way to the standard LTI system, a definition of strong observability is stated as follows:

Definition 3.1. System (2.1) with continuous output is said to be strongly observable if any initial state $x(0) \in \mathbb{R}^n$ is uniquely determined by the output $y_c(0)$ and its time derivatives $\dot{y}_c(0), \cdots, y_c^{(s)}(0)$.

Theorem 3.1. *System (2.1) is strongly observable if the following condition holds*

$$Rank[\mathcal{O}] = Rank \begin{bmatrix} C_c \\ C_cA \\ \vdots \\ C_cA^{n-1} \end{bmatrix} = n. \tag{3.1}$$

Proof. The proof is standard and it is similar to the LTI case. Just for illustration calculate at point $t = 0$ the time derivatives of the output, that is

$$\begin{aligned} y_c(0) &= C_c x_0, \\ \dot{y}_c(0) &= C_c A x_0, \\ \vdots &= \vdots \\ y_c^{(n-1)}(0) &= C_c A^{n-1} x_0. \end{aligned}$$

System (2.1) is strongly observable, if $Rank[\mathcal{O}] = n$. □

In [13, 16, 19, 21], the notion of observability proposed there allows considering measures of the output in some finite-time interval, that is, not only at the point $t = 0$, as Definition 3.1. In [16], another criterion is provided based on the matrix $[\mathcal{O}^T, G_1, \cdots, G_k]$, where G_i depends on the product $e^{At}A_I^T\mathcal{O}^T$. Here, a more concise result is provided for this case.

The following definition encompasses the similar one described in [13, 16, 19].

Definition 3.2. System (2.1) with continuous output is said to be observable on some finite-time interval $[0, t_f]$ large enough, if any initial state $x(0) \in \mathbb{R}^n$ is uniquely determined by the output $y_c(0)$, its times derivatives $\dot{y}(0), \cdots, y^{(s)}(0)$ and its time-shifts $y_c(\tau_i)$ for $\tau_i \in [0, t_f]$.

Theorem 3.2. *System (2.1) with continuous output is observable on some finite-time interval $[0, t_f]$ if and only if the following condition holds*

$$Rank \begin{bmatrix} \mathcal{O} \\ C_c e^{A\Delta_0} \\ C_c e^{A\Delta_1} A_I e^{A\Delta_0} \\ \vdots \\ C_c e^{A\Delta_{l-1}} A_I e^{A\Delta_{l-2}} \cdots A_I e^{A\Delta_1} A_I e^{A\Delta_0} \end{bmatrix} = n, \tag{3.2}$$

Proof. Consider the input $u(\tau_k) = 0 \ \forall k$ without loss of generality. The output of system (2.1) is given by $y_c(t) = C_c\Phi(\tau_k, 0)x_0$. At each time instant τ_k, $k = 1, \cdots, l$, the output measured is

$$\begin{aligned} y_c(0) &= C_c x_0, \\ y_c(\tau_1) &= C_c\Phi(\tau_1, 0)x_0 = C_c e^{A\Delta_0} x_0, \\ y_c(\tau_2) &= C_c\Phi(\tau_2, 0)x_0 = C_c e^{A\Delta_1} A_I e^{A\Delta_0} x_0, \\ \vdots &= \vdots \\ y_c(\tau_{l-1}) &= C_c\Phi(\tau_{l-1}, 0)x_0 = C_c e^{A\Delta_{l-1}} A_I e^{A\Delta_{l-2}} \cdots A_I e^{A\Delta_1} A_I e^{A\Delta_0} x_0. \end{aligned}$$

Adding the time derivatives of the output at $t = 0$, the following system

$$\begin{bmatrix} y_c(0) \\ \dot{y}_c(0) \\ \vdots \\ y_c^{(n-1)}(0) \\ y_c(\tau_1) \\ y_c(\tau_2) \\ \vdots \\ y_c(\tau_l) \end{bmatrix} = \begin{bmatrix} C_c \\ C_c A \\ \vdots \\ C_c A^{n-1} \\ C_c e^{A\Delta_0} \\ C_c e^{A\Delta_1} A_I e^{A\Delta_0} \\ \vdots \\ C_c e^{A\Delta_{l-1}} A_I e^{A\Delta_{l-2}} \cdots A_I e^{A\Delta_1} A_I e^{A\Delta_0} \end{bmatrix} x_0 \tag{3.3}$$

can be written in the form

$$\overline{y} = \mathcal{O}_{\mathcal{C}} x_0. \tag{3.4}$$

The initial condition x_0 can be uniquely determined as

$$x_0 = ({\mathcal{O}_{\mathcal{C}}}^T \mathcal{O}_{\mathcal{C}})^{-1} {\mathcal{O}_{\mathcal{C}}}^T \overline{y}. \tag{3.5}$$

System (2.1) is observable on a some finite-time interval, if $Rank[\mathcal{O}_C] = n$. □

Remark 3.1. If $A_I = dI$ with d an scalar, theorem 3.2 reduces to theorem 3.1, and system (2.1) is (strongly) observable if $Rank[\mathcal{O}] = n$.

3.2 Observability with discrete output

The notion of strong observability a linear ICS with discrete output measurements $y_d[j]$ will be understood as the ability to to retrieve x_0 before the first impulse.To develop observability over some finite-time interval, it will be assumed that impulses are applied at the same time that the output is measured.

Definition 3.3. System (2.1) with discrete output is said to be strongly observable if there exist a set of times $\{\mathcal{S} : s_0, s_1, \cdots, s_{n-1}\}$ with $s_0 = 0$ and $s_{i-1} < s_i < \infty\}$ such that any initial state $x(0) \in \mathbb{R}^n$ is uniquely determined by the output $y_d[j] = y_d(s_{j-1})$ before the first impulsive time $s_{n-1} < \tau_1 \leq s_n$.

Theorem 3.3. *the following statements are equivalent*

1. *System (2.1) with discrete output is strongly observable,*
2. $Rank \begin{bmatrix} C_d \\ C_d e^{As_1} \\ \vdots \\ C_d e^{As_{l-1}} \end{bmatrix} = n,$
3. $Rank[\mathcal{O}_d] = Rank \begin{bmatrix} C_d \\ C_d A \\ \vdots \\ C_d A^{n-1} \end{bmatrix} = n,$

Proof. 1) ⇒ 2). It follows from Eq. (2.9). The measurements of the output of system (2.1) is given by

$$y_d[j] = C_d \Phi(s_{j-1}, 0) x_0 = e^{As_{j-1}} x_0, \quad j = 1, 2, \cdots n. \tag{3.6}$$

At each measurement time instant s_j, $j = 1, \cdots, n$ the output is

$$\begin{aligned} y_d[1] &= C_d \Phi(s_0, 0) x_0 = C_d x_0, \\ y_d[2] &= C_d \Phi(s_1, 0) x_0 = C_d e^{As_1} x_0, \\ y_d[3] &= C_d \Phi(s_2, 0) x_0 = C_d e^{As_2} x_0, \\ \vdots &= \vdots \\ y_d[n] &= C_d \Phi(s_{n-1}, 0) x_0 = C e^{As_{n-1}} x_0, \end{aligned}$$

an algebraic linear system $\overline{y} = \mathcal{O}_{dI} x_0$ can be written with

$$\overline{y} = \begin{bmatrix} y_d[1] \\ y_d[2] \\ y_d[3] \\ \vdots \\ y_d[n] \end{bmatrix}, \quad and \quad \mathcal{O}_{dI} = \begin{bmatrix} C_d \\ C_d e^{As_1} \\ C_d e^{As_2} \\ \vdots \\ C_d e^{As_{n-1}} \end{bmatrix}, \tag{3.7}$$

from which x_0 can be uniquely determined as

$$x_0 = (\mathcal{O}_{dI}^T \mathcal{O}_{dI})^{-1} \mathcal{O}_{dI}^T \overline{y}. \tag{3.8}$$

System (2.1) with discrete output is strongly observable if necessarily $Rank[\mathcal{O}_{dI}] = n$.
2) $\Rightarrow$ 3) Lemma 2.3.1, and Lemma 2.3.2 from [4, pp. 30-31] yield

$$e^{A\lambda} = \sum_{i=0}^{n-1} f_i(\lambda) A^i, \quad f_0(\lambda) = 1, \quad f_{i+1}(0) = 0, \tag{3.9}$$

where $f_i(\lambda)$ is a scalar function. From the first statement, and applying the last equation, we get that

$$\begin{pmatrix} C_d \\ C_d(I + f_1(s_1)A + \cdots + f_{n-1}(s_1)A^{n-1}) \\ C_d(I + f_1(s_2)A + \cdots + f_{n-1}(s_2)A^{n-1}) \\ \vdots \\ C_d(I + f_1(s_{l-1})A + \cdots + f_{n-1}(s_{l-1})A^{n-1}) \end{pmatrix} = \Psi \mathcal{O}_d, \tag{3.10}$$

where

$$\Psi = \begin{pmatrix} I_q & 0 & \cdots & 0 \\ I_q & f_1(s_1) I_q & \cdots & f_{n-1}(s_1) I_q \\ \vdots & \vdots & \ddots & \vdots \\ I_q & f_1(s_{l-1}) I_q & \cdots & f_{n-1}(s_{l-1}) I_q \end{pmatrix} \tag{3.11}$$

and

$$\mathcal{O}_d = \begin{pmatrix} C_d \\ C_d A \\ \vdots \\ C_d A^{n-1} \end{pmatrix}. \tag{3.12}$$

The matrix Ψ has full rank if and only if

$$Q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & f_1(s_1) & \cdots & f_{n-1}(s_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & f_1(s_{l-1}) & \cdots & f_{n-1}(s_{l-1}) \end{pmatrix} \tag{3.13}$$

has full rank. From Lemma 2.3.3 in [4, pp. 33], the functions $f_i(\tau_i)$ are linearly independent in every open interval (t_s, t_{s+1}) containing the measurement time instant s_i. As it was assumed the existence of time instants $s_0, s_1, \cdots, s_{n-1}$, then Q has full rank from Lemma 2.3.3. Now, if the system is observable, necessarily $Rank[\mathcal{O}_d] = n$. □

Note this criterion is reduced to the same Kalman's criterion available for the ordinary linear time invariant discrete time system. Now, a definition for observability on a some finite-time interval will be developed. This definition fits with the definition stated in [15, 23]. Remember that it was assumed for simplicity that the measurement is done at the same time the impulse is applied to the system.

Definition 3.4. System (2.1) with discrete output is said to be observable on some finite-time interval $[0, t_f]$ large enough, if any initial state $x(0) \in \mathbb{R}^n$ is uniquely determined by discrete measurements of the output $y_d[k] = y_d(\tau_i)$ for $\tau_i \in [0, t_f]$.

Example Consider the linear impulsive system (2.1) with $\delta_i = 1$, and the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.14}$$

$$C_d = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.15}$$

To determine if the system is observable, the output is measured at different time instants τ_k, that is

- At $\tau_0 = 0$, $y[0] = x_1(0)$.
- At τ_1, $y[1] = x_1(\tau_1) = Ce^{A\delta_0}x_0 = x_1(0) + x_2(0) + \frac{1}{2}x_3(0)$.
- At τ_2, $y[2] = x_1(\tau_2) = Ce^{A}A_I e^{A}x_0 = x_1(0) + 2x_2(0) + \frac{3}{2}x_3(0)$.

From the last three equations, it is clear that the initial condition is recovered. So, this system is completely observable. However, if A_I is changed to

$$A_I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.16}$$

the system becomes unobservable for any set of time impulses. Note that the pair (C_d, A) is always observable for any matrix A_I, and the condition $\text{Rank}[C_d^T \; (C_dA)^T \; (C_dA^2)^T]^T = 2$ is not able to characterize observability.

Example Assume that a linear impulsive system is described by $\delta_i = 1$ and the same matrices C_d and B as in Example 3.2, but $A = 0$ and

$$A_I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

The discrete measurements at impulse times τ_k are

- At $\tau_0 = 0$, $y[0] = x_1(0)$.
- At τ_1, $y[1] = x_1(\tau_1) = C_d e^{A\delta_0}x_0 = x_1(0)$.
- At τ_2, $y[2] = x_1(\tau_2) = C_d e^{A}A_I e^{A}x_0 = C_d A_I x_0 = x_2(0)$.
- At τ_3, $y[3] = x_1(\tau_3) = C_d e^{A}A_I e^{A}A_I e^{A}x_0 = C_d A_I^2 x_0 = x_3(0)$.

Although $A = 0$, this linear impulsive system is observable because of the natural impulses in the state variables. For the case with $A_I = I$, the system is unobservable. Note that observability depends on the structure of the three matrices (C_d, A, A_I) and not just on the pair (C_d, A) as in the standard LTI case. Besides four measurements were needed to determined the initial condition x_0.

This set of examples suggests that the straightforward rank condition of the observability matrix involving the (C_d, A) pair only is not a complete tool to characterize observability in linear ICS with discrete output.

The observability Gramian is a known way to characterize observability of standard LTI systems. In [13], it is adapted to linear ICS with a continuous output, and in [23], with a discrete output. For the second case, it reads as: given the impulsive system (2.1) on $[0, t_f]$, and the time instants $T = \{\tau_k\}$, $k = 0, 1, 2, \cdots, l$, $t_0 = 0$, and $t_f \in [\tau_{l-1}, \tau_l)$, the observability Gramian $M_{\mathcal{O}_\mathcal{I}}(t_0, t_f)$ is defined by

$$M_{\mathcal{O}_\mathcal{I}}(0, t_f) = \sum_{j=0}^{l-1} \Phi^T(\tau_j, 0) C_d^T C_d \Phi(\tau_j, 0). \tag{3.18}$$

For the zero input and $x(0) = x_0$,

$$x_0^T M_{\mathcal{O}_\mathcal{I}}(0, t_f) x_0 = \sum_{j=0}^{l-1} \|y_d[j]\|^2, \tag{3.19}$$

from which it follows that for an observable system on $[0, t_f]$, the observability Gramian is positive definite for any impulse set time T, and any finite interval containing at least l impulse times. Conversely, if there exists an integer l such that the observability Gramian is positive definite for any impulse time set and any finite interval containing at least l impulse times, then the system is observable [23]. Note that this criterion fits to Definition 3.4.

The next theorem provides a new algebraic rank condition and asserts the equivalence between the observability Gramian criterion and this condition.

Theorem 3.4. *The following statements are equivalents*

1. *System (2.1) is observable on some finite-time interval* $[0, t_f]$,
2. $Rank \begin{bmatrix} C_d \\ C_d\Phi(\tau_1, 0) \\ \vdots \\ C_d\Phi(\tau_{l-1}, 0) \end{bmatrix} = Rank \begin{bmatrix} C_d \\ C_d e^{A\Delta_0} \\ \vdots \\ C_d e^{A\Delta_1} A_I e^{A\Delta_0} \end{bmatrix} = Rank[\mathcal{O}_I] = n,$
3. $M_{\mathcal{O}_\mathcal{I}} > 0$.

Proof. Consider the input $u(\tau_k) = 0$, $\forall k$ without loss of generality. Proof of 1) $\Leftrightarrow$ 3) can be found in [23], so it will be omitted here.

1) $\Rightarrow$ 2). It follows from Eq. (2.9). The output of system (2.1) is given by

$$y_d(\tau_k) = C_d \Phi(\tau_k, 0) x_0, \quad k = 0, 1, 2, \cdots l. \tag{3.20}$$

At each time instant τ_k, $k = 0, \cdots, l$ the output is

$$\begin{aligned} y_d(\tau_0) &= C_d\Phi(\tau_0, 0)x_0 = C_d x_0, \\ y_d(\tau_1) &= C_d\Phi(\tau_1, 0)x_0 = C_d e^{A\Delta_0} x_0, \\ y_d(\tau_2) &= C_d\Phi(\tau_2, 0)x_0 = C_d e^{A\Delta_1} A_I e^{A\Delta_0} x_0, \\ \vdots &= \vdots \\ y_d(\tau_{l-1}) &= C_d\Phi(\tau_{l-1}, 0)x_0 = C e^{A\Delta_{l-1}} \cdots A_I e^{A\Delta_0} x_0, \end{aligned}$$

the linear system above can be written as

$$\overline{y} = \begin{bmatrix} C_d \\ C_d\Phi(\tau_1, 0) \\ C_d\Phi(\tau_2, 0) \\ \vdots \\ C_d\Phi(\tau_{l-1}, 0) \end{bmatrix} x_0 = \mathcal{O}_\mathcal{I} x_0, \tag{3.21}$$

from which, if system (2.1) is observable and there exist l impulse times, then x_0 can be uniquely determined as

$$x_0 = (\mathcal{O_I}^T \mathcal{O_I})^{-1} \mathcal{O_I}^T \overline{y}, \tag{3.22}$$

and necessarily $Rank[\mathcal{O}_I] = n$.

2) ⇒ 3). If system (2.1) is observable, suppose that there is a vector $w \neq 0$ such that $w^T M_{\mathcal{O_I}} w$ is singular, *i.e.*

$$w^T M_{\mathcal{O_I}} w = w^T \left(\sum_{j=0}^{l-1} \Phi^T(\tau_j, 0) C_d^T C_d \Phi(\tau_j, 0) \right) w = 0,$$

which leads to

$$w^T M_{\mathcal{O_I}} w = \left(\sum_{j=0}^{l-1} w^T \Phi^T(\tau_j, 0) C_d^T C_d \Phi(\tau_j, 0) w \right) = 0,$$

and

$$w^T M_{\mathcal{O_I}} w = \sum_{j=0}^{l-1} \|C_d \Phi(\tau_j, 0) w\|^2 = 0.$$

From the last equation,

$$C_d \Phi(\tau_j, 0) w = 0 \quad \forall \tau_j \quad j = 0, 1, 2, \cdots \tag{3.23}$$

The latter is evaluated at l impulses applied to system (2.1) at time instant τ_j, and yields

$$\begin{bmatrix} C_d \\ C_d \Phi(\tau_1, 0) \\ C_d \Phi(\tau_2, 0) \\ \vdots \\ C_d \Phi(\tau_{l-1}, t_0) \end{bmatrix} w = \mathcal{O}_I w = 0. \tag{3.24}$$

As system (2.1) is observable, $rank[\mathcal{O}_I] = n$. In consequence, $w = 0$, which stands in contradiction and proves that $M_{\mathcal{O_I}}$ is a positive definite matrix. That completes the proof. □

Remark 3.2. Assuming that matrix $A_I = dI$, with d an scalar, System (2.1) with discrete output is (strongly) observable if and only if

$$Rank[\mathcal{O}_d] = Rank \begin{bmatrix} C_d \\ C_d A \\ \vdots \\ C_d A^{n-1} \end{bmatrix} = n. \tag{3.25}$$

4 Model of a Diabetic Type I Patient

Here, a modified Bolie's model for diabetic type I patients is considered

$$\begin{aligned} \dot{x}_1(t) &= -a_1 x_1 - a_2 x_2 + a_3 x_3, \quad x_1(0) = x_{10}, \\ \dot{x}_2(t) &= -a_4 x_2, \quad x_2(0) = x_{20}, \\ \dot{x}_3(t) &= -a_5 x_3, \quad x_3(0) = x_{30}, \\ x_2(\tau_k^+) &= x_2(\tau_k) + \frac{1}{V} u(\tau_k), \quad k \in \mathbb{N}, \\ x_3(\tau_k^+) &= x_3(\tau_k) + a_6, \quad k \in \mathbb{N}, \\ y[k] &= x_1(\tau_k) \quad k \in \mathbb{N}, \end{aligned} \tag{4.1}$$

where x_1 is the deviation of the blood glucose concentration from its basal value (assumed to be $G_b = 1.0\ g/l$), x_2 is the plasma insulin concentration in U/l, the control variable u represents a sudden change in the insulin concentration due to an injection of insulin, and y is a discrete measure of the deviation of the blood glucose concentration. The parameter a_1 is the consumption of glucose by the brain and other tissues, a_2 represents the decrease of glucose under the action of insulin, and a_4 is the natural absorption rate of insulin in the body.

Other factor that changes the concentration of the blood glucose is the input of meals. These inputs can be seen, in a day of the patient, as an impulsive jump of the glucose concentration in the body and after a digestion process as a variation in the blood concentration. As a first approximation, the digestion process is modeled as a first-order process. The variable x_3 is the glucose concentration in the body because of digestion, a_3 is the rate of variation of blood glucose concentration due to meals, and a_6 represents a sudden jump in the glucose concentration as response of the meal input.

The observability of system (4.1) is checked through Theorem 3.3. If $\delta_i = 5$ (the output is measured at each 5 min), by the statement 2),

$$Rank\,[\mathcal{O}_{\mathcal{I}}] = 3, \tag{4.2}$$

since

$$\mathcal{O}_{\mathcal{I}} = \begin{pmatrix} C_d \\ C_d e^{5A} \\ C_d e^{10A} \end{pmatrix}, \tag{4.3}$$

$a_1 \neq a_4 \neq a_5$, and

$$\det\,[\mathcal{O}_{\mathcal{I}}] = a_2 a_3 \frac{(e^{5a_5} - e^{5a_1})(e^{5a_4} - e^{5a_1})(e^{5a_5} - a_6 e^{5a_4})}{e^{10(a_1+a_4+a_5)}(a_1 - a_4)(a_1 - a_5)}. \tag{4.4}$$

The initial condition x_0 is

$$\begin{aligned} x_{10} &= y[0], \\ x_{20} &= \frac{a_2^{-1}(a_6 y[0] - (a_6 e^{5a_1} + e^{5a_5})y[1] + e^{5(a_1+a_5)}y[2])}{(a_4 - a_1)^{-1}e^{-10a_4}(e^{5a_4} - e^{5a_1})(a_6 e^{5a_4} - e^{5a_5})}, \\ x_{30} &= \frac{e^{10a_5}(y[0] - (e^{5a_1} + e^{5a_4})y[1] + e^{5(a_1+a_4)}y[2])}{a_3(a_5 - a_1)^{-1}(e^{5a_1} - e^{5a_5})(e^{5a_5} - a6e^{5a_4})}. \end{aligned} \tag{4.5}$$

To adapt the model to an individual patient the following assumptions was taken:

1. The parameter a_1, a_3, a_4 and a_5 are constants for a group of patients, and their values are taken from the literature [6, 8].
2. The parameter a_6 is given for each patient because the meals are different for each one.
3. The insulin therapy is known for all patients.
4. The glucose blood concentration is measured every 5 minutes.
5. The parameter a_2 is different for each patient and it has to be estimated.

As the system is observable, it is possible to estimate this parameter using the measurement of the output and the input, which is known. This estimation was based on clinical data. From model (4.1), the estimation formula for a_2 is

$$a_2 = \frac{a_3 x_3 - a_1 x_1 - \dot{x}_1}{x_2}. \tag{4.6}$$

The best value was obtained by a least squares method and was equal to 0.213. In Fig. 1 can be seen the response for the estimated state for the given patient. The straight line represents the real measurement of the blood glucose concentration and the dashed one the estimation of the glucose using the Bolie's model adapted to the ICS.

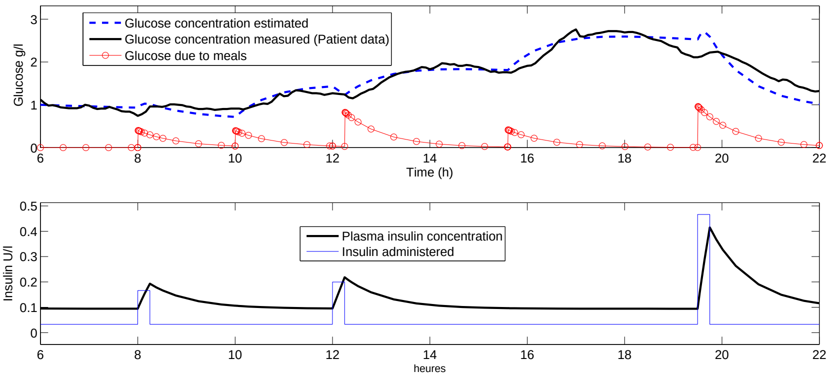

Fig. 1. Blood glucose concentration and insulin concentration estimated using data from a patient of CHU. The insulin model is a first order model

5 Conclusions and Future Works

The observability of linear time-invariant impulsive control systems with either continuous outputs or discrete outputs has been fully characterized. Criteria have been given in terms of suitable rank conditions. Throughout the paper it was shown that these criteria have to be stated in terms of the matrices (A, A_I, C). Obviously, the special case without impulses reduces to the well known Observability Kalman Criterion for linear control systems.

This result generalizes and unifies criteria that can be found in the current literature, using the observability Gramian and algebraic conditions.

Future research challenges include the generalization to switched impulsive systems, and the design of effective impulsive observers for biological applications.

Parameter identifiability and the effective identification are connected problems which are worth to investigate within a similar approach. Its application to parameter identification of the glycemia dynamics is promising for diagnosis for patients. The latter is open for further research as well.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Choi Y, Mok S, Bang H. Impulsive formation control using orbital energy and angular momentum vector. Acta Astronautica. 2010;67(5-6):613-622.

[2] Rivadeneira P, Moog C. Impulsive control of single-input nonlinear systems with application to hiv dynamics. Applied Mathematics and Computation. 2012;218(17):8462-8474.

[3] Sun J. Impulsive control of a new chaotic system. Mathematics and Computers in Simulation. 2004;64(6):669-677.

[4] Yang T. Impulsive Control Theory, 1st Edition, LNCIS, Springer-Verlag Berlin Heidelberg; 2001.

[5] Yu H, Zhong S, Agarwal RP. Mathematics analysis and chaos in an ecological model with an impulsive control strategy. Communications in Nonlinear Science and Numerical Simulation. 2011;16(2):776-786.

[6] Bolie V. Coefficients of normal blood glucose regulation. Journal of Applied Physiology. 1961;16(5):783-788.

[7] Bergman R, Phillips L, Cobelli C. Physiologic evaluation of factors controlling glucose tolerance in man: Measurement of insulin sensitivity and beta-cell glucose sensitivity from the response to intravenous glucose. Journal of Clinical Investigation. 1981;68(6):1456-1467.

[8] Cobelli C, Mari A. Validation of mathematical model of complex endocrine-metabolic systems. Medical & Biological Engineering & Computing. 1983;21(4):390-399.

[9] Gaetano A, Arino O. Mathematical modelling of the intravenous glucose tolerance test. Journal of Mathematical Biology. 2000;40(2):136-168.

[10] Magdelaine N, Chaillous L, et al. A long-term model of the glucose–insulin dynamics of type 1 diabetes. IEEE Transactions on Biomedical Engineering. 2015;62(6):1546-52.

[11] Ruan Y, et al. Modeling day-to-day variability of glucose–insulin regulation over 12-week home use of closed-loop insulin delivery. IEEE Transactions on Biomedical Engineering. 2017;64(6):1412-1419.

[12] Lakshmikantham V, Bainov D, Simeonov PS. Theory of impulsive differential Equations. Series in Modern Applied Mathematics, World Scientific, Singapore. 1989;6.

[13] Guan Z, Qian T, Yu X. On controllability and observability for a class of impulsive systems. Systems & Control Letters. 2002;47(9):247-257.

[14] Liu X. Impulsive control and optimization. Applied Mathematics and Computation. 1995;73:77-98.

[15] Medina E, Lawrence D. Reachability and observability of linear impulsive systems. Automatica. 2008;44:1304-1309.

[16] Shi H, Xie G. Controllability and observability criteria for linear piecewise constant impulsive systems. Journal of Applied Mathematics. Article ID 182040. 2012;24.

[17] Zhao S, Sun J. A geometric approach for reachability and observability of linear switched impulsive systems. Nonlinear Analysis: Real World Applications. 2010;72:4221-4229.

[18] Rivadeneira PS, Moog CH. Observability criteria for impulsive control systems with applications to biomedical engineering processes. Automatica. 2015;55:125-131.

[19] Zhao S, Sun J. Controllability and observability for a class of time-varying impulsive systems. Nonlinear Analysis: Real World Applications. 2009;10:1370-1380.

[20] Fang T, Sun J. Controllability and observability of complex [r]-matrix time-varying impulsive systems. Advances in Difference Equations. 2013;1:129-145.

[21] Xie G, Wang L. Controllability and observability of a class of linear impulsive systems. Journal of Mathematical Analysis and Applications. 2005;304:336-355.

[22] Zhao S, Sun J. Controllability and observability for impulsive systems in complex fields. Nonlinear Analysis: Real World Applications. 2010;11:1513-1521.

[23] Medina E, Lawrence D. State estimation for linear impulsive system, in: Proceedings of the American Control Conference, Hyatt Regency Riverfront, St. Louis, MO, USA. 2009;1183-1188.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/22177