**Journal of Advances in Mathematics and Computer Science**

**26(3): 1-10, 2018; Article no.JAMCS.38568**

***ISSN: 2456-9968***

*(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)*

# Root Systems, Cartan Matrix and Dynkin Diagrams in Classification of Lie Algebras

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***Authors' contributions***

*This work was carried out in collaboration between the two authors. They all read and approved the final version of their manuscript.*

***Article Information***

DOI: 10.9734/JAMCS/2018/38568
Editor(s):
(1) Morteza Seddighin, Professor, Indiana University East Richmond, USA.
(2) Dragos-Patru Covei, Professor, Department of Applied Mathematics, The Bucharest University of Economic Studies, Piata Romana, Romania.
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(5) Emanuel Guariglia, University of Salerno, Italy.
Complete Peer review History: http://www.sciencedomain.org/review-history/23152

**Review Article**

***Received: 1st November 2017***
***Accepted: 1st February 2018***
***Published: 12th February 2018***

## Abstract

This review paper deals with Lie algebras, with some concentration on root systems, which help in classification and many applications of symmetric spaces. We deal with the basic concept of a root system. First, its origins in the theory of Lie algebras are exposed, then an axiomatic definition is provided. Bases, Weyl groups, and the transitive action of the latter on the former are explained. Finally, the Cartan matrix and Dynkin diagram are exposed to suggest the multiple applications of root systems to other fields of study and their classification.

*Keywords: Lie algebras; Root systems; Weyl group; Cartan Matrix and Dynkin diagrams.*

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# 1 Introduction

In our discussion of root systems, we begin with a general overview of their role in the theory of Lie algebras. A Lie algebra is a vector space endowed with a bilinear operation known as the bracket operation [,], defined for all elements of the Lie algebra and satisfying certain properties. A Lie algebra is called simple if it has no proper ideal, and it is semisimple if it is a direct sum of simple ideals. Also for the semisimple Lie algebra L, we define a toral subalgebra as the span of some semisimple elements of $L$ . For a maximal toral subalgebra $H$ which is not properly contained in any other, $L$ may then be written as the direct sum of $H$ and the subspaces $L_\alpha = \{x \in L \setminus [h, x] = \alpha(h)x, \forall h \in H\}$ where $\alpha$ ranges over all elements of $H^*$ ( the dual space of H). In this respect, the nonzero $\alpha$ for which $L_\alpha \neq 0$ are called the roots of $L$ relative to $H$. These Roots, form what is called root systems which provide a convenient way for completely characterizing simple and semi simple Lie algebras. The goal of this review paper is to disclose some relations and results related to root systems, Cartan matrices, Dynkin diagrams and finally discussing the problem of classification of Lie algebras, [1,2,3,4,5,6,7].

# 2 Lie Algebra

To study a Lie algebra, we must know that it is a linearization of its original Lie group, so one remembers that a Lie group is a group provided that its two operations: multiplication and inversion are smooth maps.

A Lie algebra is an algebraic structure whose main use is in studying geometric objects such as Lie groups and differentiable manifolds.

## 2.1 Definition

A Lie algebra is a pair $(V, [\,,])$ where $V$ is a vector space ,and [ , ] is a Lie bracket, $[\,,\,]: V \times V \to V$ satisfying:

(1) $[v, w] = -[w, v]$ skew-symmetric .
(2) $[av + bu, w] = a[v, w] + b[u, w]$ , bilinearity.
(3) $[v, [w, u]] + [w, [u, v]] + [u, [v, w]] = 0,$
for all $u, v$ and $w \in V$. (Jacobi identity ).

A Lie Bracket is a binary operation [ , ] on a vector space.

In fact the Lie bracket for $X, Y \in V$ , is defined by $[X, Y] = XY - YX$ , which is in some special cases is called the commutator.

## 2.2 Example

The Lie algebra $\boldsymbol{g}$ of $R^n$ as a Lie group ,is again $R^n$where

$$[X, Y] = 0, \forall X, Y \in G .$$

Thus the Lie bracket for the Lie algebra of any abelian group is zero, that is as in the commutator mentioned in 2.1 above.

## 2.3 Example

A homeomorphism of Lie algebra $\ell$ is a linear map, $\varphi : \ell \to \ell$ , preserving the Lie bracket . This means that $\varphi[\ell_1, \ell_2] = [\varphi(\ell_1), \varphi(\ell_2)]$ for any $(\ell_1, \ell_2) \in \ell \times \ell$ .

Proof:

$$\varphi[\ell_1,\ell_2]=\varphi[\ell_1\ell_2-\ell_2\ell_1]=[\varphi(\ell_1)\varphi(\ell_2)-\varphi(\ell_2)\varphi(\ell_1)]$$
$$=[\varphi(\ell_1),\varphi(\ell_2)] ,$$

which shows the claimed linearity and preserving the bracket in $\ell$. It worth mentioning that the homomorphism in this example is the same as homomorphism defined between groups as general, and as we know, the Lie algebra of a Lie group can be seen as the vector space at the identity element of its Lie group.

### 2.4 Definition

A vector subspace $\eta$ of a Lie algebra $\ell$ is called a **Lie subalgebra** if $[\eta,\eta] \subseteq \eta$ .

An **ideal** of $\ell$ is a Lie subalgebra $\eta$ such that $[\eta,\ell] \subseteq \eta$.

### 2.5 Theorem[2]

Let $G$ be a Lie group and $L$ its Lie algebra

(1) If $H$ is a Lie subgroup of G, $\eta$ is a Lie subalgebra of $L$.
(2) If $\eta$ is a Lie subalgebra, there exists a unique Lie subgroup $H$ of $G$ such that the Lie algebra of $H$ is isomorphic to $\eta$.

### 2.6 Some properties of Lie algebras

Lie algebras have many properties related to their effects on linear operators. As we know a Lie algebra is a linearization of its Lie group, that is, a Lie algebra is a linear vector space in which linear operations can be carried on it easily, rather than on its original Lie algebra. So we give some properties of Lie algebras related to linear operators below:

(i) The operators in a Lie algebra form a linear vector space.
(ii) The operators closed under commutation: the commutator of two operators is in the Lie algebra;
(iii) The operators satisfy the Jacobi identity.

## 3 Root Systems

### 3.1 Roots (introduction)

The root or root vectors of a Lie algebra are the weight vectors of its adjoint representation. Roots are very important because they can be used both to define Lie algebra and to build their representations. We will see that Dynkin diagrams are in fact really only a way to encode information about roots. The number of roots is equal to the dimension of Lie algebra which is also equal to the dimension of the adjoint representation , therefore we can associate a root to every element of the Lie algebra. The most important things about roots is that they allow us to move from one weight to another. (weights are vectors which contain the eigenvalues of elements of Cartan subalgebra), for more details , see [8,9,10].

### 3.2 Definition (root systems)

Let $V$ be a real finite-dimensional vector space and $R \subset V$ a finite set of nonzero vectors , $R$ is called a root systems in $V$ (and its members are called roots) if :

(i) $R$ generates $V$.
(ii) For each $\propto \in R$ there exists a reflection $S_\propto$ along $\propto$ leaving $R$ invariant.

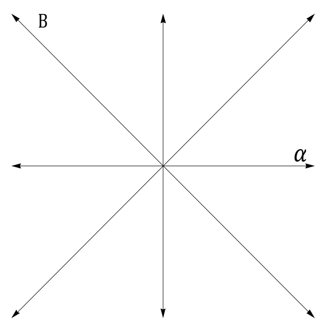
(iii) For all $\propto, \beta \in R$ the number $a_{\mathcal{B},\propto}$ determined by $S_\alpha \beta = \beta - a_{\beta\alpha}\alpha$ is an integer, that is $a_{\mathcal{B},\propto} \in \mathbb{Z}$ (the set of integers).

### 3.3 Theorem[11]

Every root system has a set of simple roots $\Delta$, such that each $\alpha \in \Delta$ may be written as $\alpha = \sum_{\delta \in \Delta} K_\delta \delta$, with $K_\delta \in \mathbb{Z}$ and each $K_\delta$ has same sign.

### 3.4 Example

The following example is of the root system $B_2$. Where α and β below form a base for $B_2$.




The roots of a given basis are called simple.

### 3.5 Definition (positive roots)

The set of positive roots of a root system $\Phi$ is denoted by $\Phi^+$, where $\Phi^+ \subset \Phi$, and these positive roots satisfy:

1) For all $\alpha \in \Phi$, only $\alpha$ or $-\alpha$ is in $\Phi$.
2) If $\alpha, \beta \in \Phi^+$, and $\alpha + \beta$ is a root, then $\alpha + \beta \in \Phi$.

### 3.6 Definition (simple roots)

We call $\alpha \in \Phi^+$ a simple root for $\Phi$ if it is not a sum of two other positive roots. The set of simple roots is denoted by $\Delta$ (or sometimes by $\Pi$).

The number of simple roots, is equal to the dimension of the Euclidean space $E$. Also a root in $\Delta$ is called indecomposable if it can't be written as a linear combination of any other roots . By selecting all the indecomposable roots whose inner product with a predetermined vector $y$ in $E$ is positive, one obtains a set of linearly independent roots α which lie entirely on the same side of the hyperplane normal to $y$. Then –α is not contained in the set of all α, and in fact these roots both span $\mathbb{R}^n$ and give rise to all other roots, see [12,3].

### 3.7 Definition (reduced root systems)

Let $V$ be an Euclidean vector space [finite–dimensional real vector space with the canonical inner product $(.,.)$ ]. Then $R \subseteq V \setminus \{0\}$ is a reduced root systems if it has the following properties :

(1) The set $R$ is finite and it contains a basis of the vector space $V$.
(2) For roots $\alpha, \beta \in R$ we demand $n_{\alpha,\beta}$ to be integer :

$$n_{\alpha,\beta} \equiv \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z} \; .$$

(3) If $S_\alpha : V \to V$ is defined by $S_\alpha(\lambda) = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha$. Then for any two roots $\alpha, \beta \in R$, we have $S_\alpha(\beta) \in R$.
(4) If $\alpha, c\alpha \in R$ for some real $c$, then $c = 1 \quad or -1$.

### 3.8 Remarks

If $\alpha, \beta \in R$ are proportional, $\beta = m\alpha$ ( $m \in \mathbb{R}$ ) , then $m = \pm\frac{1}{2}, \pm 1, \pm 2$.

In fact the numbers $a_{\alpha, m_\alpha} = 2/m$ and $a_{m_\alpha, \alpha} = 2m$ are both integers. A root system $R$ is said to be reduced if $\alpha, \beta \in R$, $\beta = m\alpha$ implies $m = \pm 1$. A root $\alpha \in R$ is called indivisible if $\frac{1}{2}\alpha \notin R$ and unmultipliable if $2\alpha \notin R$.

### 3.9 Example

(i) The set $\Delta = \Delta(g, \mathfrak{h},)$ of roots of a semisimple Lie algebra $\boldsymbol{g}$ over $\mathbb{C}$ with respect to a Cartan subalgebra $\mathfrak{h}$ is a reduced root system.
(ii) The set $\sum$ $of$ restricted roots is a root system which in general is not reduced.

Restricted root systems are sometimes called relative root systems, and they are those related to symmetric spaces. To calculate them, we find an alternative Cartan subalgebra, which will not be discussed in this short paper, but the reader can see [3].

## 4 Weyl Group

### 4.1 Definition (Weyl group)

The Weyl group is the group generated by the reflections $S_\alpha$, $\alpha \in \Delta$ . Where $S_\alpha$ is the reflection generated by the root $\alpha$.

So we see that Weyl group $W$ of a root system consists of all the reflections $\sigma_\alpha$ generated by elements $\alpha$ of the root system. For a given root α, the reflection $\sigma_\alpha$ fixes the hyperplane normal to $\alpha$ and maps $\alpha \to -\alpha$. We may write $\sigma_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \alpha$ . Where $\langle \alpha, \beta \rangle = n_{\alpha,\beta}$ . The hyperplanes fixed by the elements of $W$ partition $E$ into Weyl chambers. For a given base $\Delta$ of $E$, the unique Weyl chamber containing all vectors $y$ such that $(y, \alpha) \geq 0$ , $\forall \alpha \in \Delta$ is called the fundamental Weyl chamber. $W^{/}$is the subgroup of $W$

generated by only those rotations arising from the simple roots of a given base. See [13,12,3,7,14]. We have the following theorems explaining some properties of $W$ and $W^{/}$:

### 4.2 Theorem[7]

Given $\Delta$ and $\Delta^{/}$ two bases of a root system $\Phi$, then $\Delta^{/} = \sigma(\Delta)$ for some $\sigma \in W^{/}$.

### 4.3 Lemma[7]

For all $\alpha \in \phi$, $\exists\ \sigma \in W$ such that $\sigma(\alpha) \in \Delta$.

### 4.4 Lemma[7]

$W^{/} = \{ \sigma_\alpha$ arising from $\alpha \in \Delta\}$ generates W.

### 4.5 Theorem[2]

(i) Each root system has a basis.
(ii) Any two bases are conjugate under a unique weyl group element.
(iii) $n_{\alpha,\beta} \leq 0$ for any two different element $\alpha, \beta$ in the same basis.

## 5 The Cartan Matrix

To describe relative position of roots $\alpha_i \in \Delta$, we find all inner products between these roots. Although this is not invariant under isomorphisms of root systems, we use what is called Cartan matrix for describing relative positions of simple roots. It is not a symmetric matrix.

### 5.1 Definition (Cartan matrix)

The Cartan matrix of a set of simple roots is an n- square matrix with entries

$$\alpha_{ij} = \langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

### 5.2 Definition (The generalized Cartan matrix)

A generalized Cartan matrix $A = (a_{ij})$ is a square matrix with integral entries such that:

(1) For non-diagonal entries, $a_{ij} \leq 0$.
(2) $a_{ij} = 0$ if and only if $a_{ji} = 0$
(3) $A$ can be written as $DS$ where $D$ is a diagonal matrix and $S$ is a symmetric matrix.

More properties of Cartan matrix follows from the following lemma:

### 5.3 Lemma [15]

(1) For any $i, a_{ii} = 2.$

(2) For any $i \neq j, a_{ij}$ is a non-positive integer: $a_{ij} \in \mathbb{Z}, a_{ij} \leq 0$.

(3) For any $i \neq j, a_{ij}a_{ji} = 4\cos^2 \varphi$, where $\varphi$ is the angle between $a_i \,\&\, a_j$. If $\varphi \neq \pi/2$, then $\dfrac{|\alpha_i|^2}{|\alpha_j|^2} = \dfrac{a_{ji}}{a_{ij}}$.

The following theorem shows the role of Cartan matrices in determination of root systems:

## 5.4 Theorem [11]

The Cartan matrix determines the root system $\Phi$ up to isomorphism.

## 5.5 Example

For the root system $B_2$ introduced previously, its Cartan matrix has the following form:

$$A = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}.$$

## 5.6 Example

For the root system $A_n$, we can find the Cartan matrix as :

$$A = \begin{bmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \vdots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

, where entries not shown in the matrix are zeros.

## 5.7 Theorem[7]

Given two root systems $\phi \subset E$ and $\phi' \subset E'$ with bases

$\Delta = \{\alpha_i, \alpha_j, \dots, \alpha_l\}$ and $\Delta' = \{\alpha_i', \alpha_j', \dots, \alpha_l'\}$ with identical Cartan matrices i.e $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_i', \alpha_j' \rangle$ for $1 \leq i, j \leq n$. Then this bijection extends to an isomorphism $f: E \longrightarrow E$ which maps

$$\phi \longrightarrow \phi' \ \ and\ satisfies\ \langle f(\alpha), f(\beta) \rangle = \langle \alpha, \beta \rangle\ , \forall\ \alpha, \beta \in \phi.$$

Note that: information contained in Cartan matrix can be presented graphically by using Dynkin diagrams, given below:

# 6 Dynkin Diagrams

Dynkin diagrams are simple methods allowing us to classify all simple Lie algebras. They contain a set of nodes (vertices) whose number is equivalent to the rank of the Lie algebra( the number of simple roots).Each of these nodes corresponds to a particular simple root.

Irreducible root systems provide a simple means of classifying Lie algebras. However, the root systems may themselves be classified according to their Dynkin diagrams. Each such diagram belongs to one of finitely many families of graphs with a variety of connections to e.g. quiver representations. Correspondence between Cartan matrices and Dynkin diagrams may be explicitly understood as follows:

Each vertex of the Dynkin diagram corresponds to a root $\alpha_i$. Clearly if $C_{ij}$=$C_{ji}$= 0**,** no edge exists between the vertices for $\alpha_i$ and $\alpha_j$. If the $C_{ij}$th and $C_{ji}$th entries in the Cartan matrix are both ±1, a single edge connects the vertices corresponding to $\alpha_i$ *and* $\alpha_j$. If the $C_{ij}$th or $C_{ji}$th entry is ±2 or ±3, two or three edges ,respectively ,connect the two vertices in question. In order to distinguish the relative lengths of the roots, an arrow pointing towards the shorter of the two is drawn over the vertex in question .The properties of the Cartan matrices place a number of restrictions on possible Dynkin diagrams, which we enumerate below. In fact, these properties, enumerated below, lead to a complete description of all possible Dynkin diagrams, which may be found in [11,16,12,7].

1) If some of the vertices of the Dynkin diagram are omitted along with all their attached edges, the remaining graph is also possible as a Dynkin diagram.
2) The number of vertex pairs connected by at least one edge is strictly less than the order of the root system. It follows that no Dynkin diagram may contain a cycle.
3) No more than three edges can connect to a single vertex. Thus, the only Dynkin diagrams containing a triple edge contain exactly those two vertices it connects.
4) If a Dynkin diagram contains as a subgraph a simple chain, the graph obtained by reducing that chain to a point also forms a Dynkin diagram. This prohibits several possible arrangements of terminal vertices from co-occurring within a diagram, lest the preceding restriction be violated, see [7].

## 6.1 Definition (Dynkin diagram)

Let $\Delta$ be the set of simple roots of a root system $\Phi$, then we construct the Dynkin diagram of $\Delta$ in the following description:

- Each simple root $\alpha_i$ is represented by a circle or a dot as a vertex in Dynkin diagram.
- For each pair of simple roots $\alpha_i \neq \alpha_j$, we connect the corresponding vertices by n edges, where n depends on the angle $\varphi$ between the two roots:

  For $\varphi = \pi/2, n = 0$, the vertices are not connected ,and the case is $A_1 \times A_1$ diagram.

  For $\varphi = 2\pi/3, n = 1$, the case is $A_2$ diagram with a single edge.

  For $\varphi = 3\pi/4, n = 2$, the case is $B_2$ diagram with a double edge.

For $\varphi = 5\pi/6, n = 3$, the case is $G_2$ diagram with a triple edge.

- If $\alpha_i \neq \alpha_j$, $|\alpha_i| \neq |\alpha_j|$ and $(\alpha_i, \alpha_j) \neq 0$, we orient the corresponding (multiple) edge by putting on it an arrow pointing towards the shorter root.

### 6.2 Example

For rank two root systems, see their Dynkin diagrams below:


All cases of Dynkin diagrams and the complete classification of Lie algebras can be seen in [12,3].

By now we reach some general results which pave the way for researchers to carry their job using the fundamental references [12,3] and other references. In addition, the problem of classification of symmetric spaces using the root systems of Lie algebras can be carried for more results and applications in other scientific fields.

## 7 Some Conclusions

- Lie algebras can be used in classification of some spaces, especially symmetric spaces, because they can be used as algebraic approaches to these spaces.
- We can have several different spaces derived from the same Lie algebra.
- Many properties of what is called symmetric spaces can be studied through their Lie algebras and root systems and this can help in extracting many of their properties.
- Root systems, Dynkin diagrams and Cartan matrix play an important role in classification of Lie algebras and they simplify this job.
- It is possible to reconstruct the root system from its Cartan matrix.
- A finite root system can be encoded by Cartan matrix, which in turn can be encoded even more compactly by a Dynkin diagram.
- Dynkin diagrams correspond bijectively with finite-dimensional simple complex Lie algebras, and therefore the classification of Dynkin diagrams is actually a classification of all such Lie algebras.

## Competing Interests

Authors have declared that no competing interests exist.

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