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# Reviewing Stability Criteria for Positive Homogeneous Systems and Adding One for Discrete-Time Cases with Degree Less Than One

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## Abstract

The article synthesizes and presents the results regarding the stability of positive homogeneous systems that have been researched and published in recent years. Next, we provide a sufficient condition for global exponential stability in the case of discrete-time positive homogeneous systems with an order less than one with time-varying delays.

## Keywords

Exponential Stability, Homogeneous Systems, Positive Systems, Positive Homogeneous Systems

## 1. Introduction

Positive dynamical systems are systems in which their state variables remain non-negative at all times in the future, given non-negative initial conditions. This class of systems finds numerous applications in various fields of science, engineering, and economics, such as biology, chemistry, communication systems, economics, and more [1] [2].

Time delay is a prevalent occurrence in most dynamical systems, particularly in technical and information-related fields such as mechanical control, electronics, and telecommunications [1] [2] [3]. It stands as one of the factors impacting system stability, reducing operational performance. In general, time delay has a negative impact on the system and needs to be taken into consideration when analyzing the stability of dynamical systems. Linear systems have been studied extensively and have yielded rich results from early on. The class of positive linear systems has also been actively researched in recent years by mathematicians [4] [5] [6] [7]. Various stability criteria for this class of systems have been proposed, including necessary and sufficient conditions for positive linear systems without delays and with constant delays. A notable result is that the stability of positive linear systems without delays and with constant delays is equivalent, meaning that stability does not depend on the delay. This result has also been extended to positive linear systems with time-varying delays.

There is a truth that important real-world systems are often nonlinear. Therefore, a natural question arises: do the non-sensitive properties of linear positive systems with respect to time delay still hold for nonlinear positive systems? In [8], a specific class of nonlinear positive systems with these properties was identified, namely, the class of homogeneous cooperative positive systems. Systems without delays or with constant delays are indeed an ideal scenario because delays typically depend on time. Specifically, the authors in [9] demonstrated that the global asymptotic stability of homogeneous cooperative systems is independent of a constant delay, meaning that homogeneous cooperative systems with a constant delay are globally asymptotically stable if and only if the corresponding delay-free systems are globally asymptotically stable. This result is purely qualitative and does not provide any quantitative information, such as decay rates in exponential functions, polynomial functions, or exponential functions. Therefore, there is relatively limited research on this class of systems at this point, and one of the reasons is that Lyapunov-Krasovskii function techniques, while very useful for stability analysis in classes of delay-free positive systems or those with constant delays, cannot be applied to classes of positive systems with time-varying delays or lead to overly conservative results.

The stability analysis of a positive system with global asymptotic stability is inherently qualitative, whereas practical requirements often demand a quantitative assessment of system stability. An evaluation of the rate of decay in exponential functions (for discrete-time systems) or in polynomial and exponential functions (for continuous-time systems) provides us with more information, such as real-time state bounds, finite-time constraints, predefined time horizons, ultimate bounds, reachable sets, invariant bounded sets, etc. [10]. In recent years, there have been numerous studies and quantitative findings on the stability of the class of homogeneous cooperative systems with delays. By developing a novel evaluation technique, Feyz and colleagues published various results in [11] and synthesized and presented them in the thesis "Performance Analysis of Positive Systems and Optimization Algorithms with Time-delays" in 2014 [12]. Although Feyz has established global exponential stability criteria for homogeneous cooperative systems with delays in both continuous and discrete-time cases, these criteria are limited to homogeneous systems of degree one. Inspired by the techniques developed by Feyz, Dong conducted research on the class of cooperative systems with arbitrary degrees of homogeneity and time-varying delays. In [13], through the analysis of cases regarding the degree of homogeneous vector fields, Dong achieved certain results such as 1) providing necessary and sufficient conditions for global polynomial stability of non-linear positive systems (continuous-time) and establishing local exponential stability (discrete-time) with delays dependent on time when the vector field is cooperatively homogeneous with a degree greater than one, and 2) for vector fields with a degree less than one, giving necessary and sufficient conditions for finite-time global stability of cooperative homogeneous systems without delay (continuous-time case). Dong's results are independent of the system's delay but are contingent upon the corresponding vector fields. However, Dong still leaves several cases unexplored, such as global exponential stability for the class of positive systems with time-varying delays and vector fields that are cooperatively homogeneous of degree less than one (both in continuous-time and discrete-time cases). Recognizing the limited research outcomes for the case of positive systems with vector fields cooperatively homogeneous of degree less than one, Q. Xiao and colleagues in [14] delved deeper into this scenario and achieved notable results, such as 1) analyzing and extending results from [12] regarding the global exponential stability of the class of positive homogeneous systems with time-varying delays in the case of vector fields having a degree less than one and 2) addressing one of the missing results from [13], which is establishing a finite-time stability criterion for delay-free systems with vector fields that are cooperatively homogeneous of degree less than one, where the stability time interval and an upper bound for the states are determined. To the best of our knowledge, Q. Xiao's findings are the most recent for the class of positive cooperative homogeneous systems, and Q. Xiao has yet to resolve the case of global exponential stability for the class of positive systems with time-varying delays and vector fields that are cooperatively homogeneous of degree less than one in discrete-time.

In addition to reviewing the results previously researched and published by other authors, our article presents a sufficient condition for assessing global exponential stability for the class of positive cooperative systems with time-varying delays, where the vector fields are cooperatively homogeneous of degree less than one in discrete-time.

## 2. Notations, Model, and Preliminary

### 2.1. Notations

Recall that $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n_+}$ denote the set of real numbers, n-dimensional real vector space, and non-negative n-dimensional real vector space, correspondingly. For $x \in \mathbb{R}$, let $\lceil x \rceil$ be the smallest integer not less than *x*. For an arbitrary vector $x \in \mathbb{R}^n$, $x_i$ is its *i*-th component. Given two vectors $x, y \in \mathbb{R}^n$, we write: $x \geq y$ ($x > y$) if and only if $x_i \geq y_i$ ($x_i > y_i$) for all $1 \leq i \leq n$, and *x* is said to be positive if $x > \mathbf{0}$, with $\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n$. Given a positive vector $v \in \mathbb{R}^n$, the weighted $l_\infty$ norm of a vector $x \in \mathbb{R}^n$ is defined by

$$\|x\|_\infty^v = \max_{1 \leq i \leq n} \frac{|x_i|}{v_i}.$$

A matrix $A = \left[a_{ij}\right] \in \mathbb{R}^{n \times n}$ is said to be nonnegative if $a_{ij} \geq 0$ for $1 \leq i, j \leq n$. The matrix *A* is said to be Metzler if it satisfies $a_{ij} \geq 0$ for every $i \neq j$. Give a real interval $[a, b]$, $\mathcal{C}\left([a, b], \mathbb{R}^n\right)$ denotes the space of all real-valued continous functions on $[a,b]$ taking values in $\mathbb{R}^n$.

### 2.2. Model

We consider the nonlinear homogeneous system with time-varying delays

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t-\tau(t))), & t \ge 0 \\ x(s) = \varphi(s), & s \in [-\tau_{\max}, 0] \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector of system, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are the characterizes vector fields for nonlinear dynamical systems, the delay function $\tau(t)$ is continuous and is such that $0 \le \tau(t) \le \tau_{\max}$ for $t \ge 0$ and $\varphi(t) \in \mathcal{C}([-\tau_{\max}, 0], \mathbb{R}_+^n)$.

The discrete-time analog (1) takes the form

$$\begin{cases} x(k+1) = f(x(k)) + g(x(k-d(k))), & k \in \mathbb{Z}_+ \\ x(s) = \varphi(s), & s \in \{-d_{\max}, \cdots, 0\} \end{cases} \quad (2)$$

### 2.3. Preliminary

Here, we introduce the fundamental concepts regarding cooperative, homogeneous, and order-preserving vector fields. These definitions and results can be referenced in detail in [2].

**Definition 1.1.** *System* (1) *is called positive if for any non-negative initial conditions, its state trajectory* $x(t)$ *satisfies* $x(t) \ge 0$ *for all* $t \ge 0$.

**Definition 2.2.** *A continuous vector field* $f : \mathbb{R}^n \to \mathbb{R}^n$ *which is continuously differentiable on* $\mathbb{R}^n \setminus \{\mathbf{0}\}$ *is called cooperative if the Jacobian matrix* $\frac{\partial f}{\partial x}(a)$ *is Metlzer for all* $a \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$.

**Definition 2.3.** *A vector field* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is called homogeneous of degree* $\alpha > 0$ *if* $f(\lambda x) = \lambda^\alpha f(x)$ *for all* $x \in \mathbb{R}^n$ *and all* $\lambda > 0$.

**Definition 2.4.** *A vector field* $g : \mathbb{R}^n \to \mathbb{R}^n$ *is called order-preserving on* $\mathbb{R}_+^n$ *if* $g(x) \ge g(y)$ *for any* $x, y \in \mathbb{R}_+^n$ *such that* $x \ge y$.

**Lemma 2.5.** *Let f be a cooperative vector field, then for any* $x, y \in \mathbb{R}_+^n \setminus \{0\}$ *satisfying* $x \ge y$ *and* $x_i = y_i, 1 \le i \le n$, *we have* $f_i(x) \ge f_i(y)$.

The research findings and results presented in the following section are obtained by imposing the following assumptions on system (1).

**Assumption 1.** *f* and *g* are continuous on $\mathbb{R}^n$ and continuously differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, and homogeneous of degree $\alpha > 0$.

**Assumption 2.** *f* is cooperative and *g* is order-preserving on $\mathbb{R}_+^n$.

Assumption 1 ensures the existence and uniqueness of solutions for system (1) for all non-zero initial functions [15] and Assumption 2 ensures its positively [2].

## 3. Review of Results

In this section, we review the results that have been published in recent years. The results are organized by the degree of the homogeneous vector field.

### 3.1. The Case of Degree $\alpha = 1$

By developing a novel technique, Feyz, in [12], has achieved preliminary results in quantifying the stability of the class of cooperative positive systems with homogeneous vector fields of degree one.

**Theorem 3.1 (Continuous-time case).** *Consider the system* (1) *that satisfies Assumption* 1 *and Assumption* 2. *The following statements are equivalent*:

i) *There exists a vector* $v > 0$ *such that*

$$f(v) + g(v) < \mathbf{0}.$$

ii) *System* (1) *is globally exponentially stable for all bounded time delays. Specifically, every solution* $x(t)$ *of* (1) *satisfies*

$$\|x(t)\|_\infty^v \le \|\varphi\| \mathrm{e}^{-\eta t}, \quad t \ge 0,$$

*where* $\|\varphi\| = \max_{-\tau_{\max} \le s \le 0} \|\varphi(s)\|_\infty^v$, $\eta \in \left(0, \min_{1 \le i \le n} \eta_i\right)$, *and* $\eta_i$ *is the unique positive solution of the equation*

$$\frac{f_i(v)}{v_i} + \frac{g_i(v)}{v_i} \mathrm{e}^{\eta_i \tau_{\max}} + \eta_i = 0, \quad i = 1, 2, \cdots, n.$$

This result can be readily extended to the case of multiple delays of the form

$$\dot{x}(t) = f(x(t)) + \sum_{k=1}^{p} g\left(x\left(t - \tau_k(t)\right)\right).$$

In this case, the exponential stability condition is

$$f(v) + \sum_{k=1}^{p} g(v) < \mathbf{0}.$$

A specific instance of system (1) with homogeneous vectors *f* and *g* of degree one corresponds to continuous-time linear systems in the following form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx\left(t - \tau(t)\right), & t \ge 0 \\ x(s) = \varphi(s), & s \in [-\tau_{\max}, 0] \end{cases} \tag{3}$$

**Theorem 3.2. (Continuous-time case).** *Consider the system* (3) *that satisfies A is Metlzer and B is nonnegative. The following statements are equivalent*:

i) *There exists a vector* $v > 0$ *such that*

$$(A + B)v < \mathbf{0}.$$

ii) *System* (3) *is globally exponentially stable for all bounded time delays.*

**Theorem 3.3. (Discrete-time case).** *Consider the system* (2) *that satisfies Assumption* 1 *and Assumption* 2. *The following statements are equivalent*:

i) *There exists a vector* $v > 0$ *such that*

$$f(v) + g(v) < v.$$

ii) *System* (2) *is globally exponentially stable for all bounded time delays. Specifically, every solution* $x(t)$ *of* (2) *satisfies*

$$\|x(k)\|_\infty^v \le \|\varphi\| \gamma^k, \quad k \in \mathbb{Z}_+,$$

*where* $\|\varphi\| = \max_{-d_{\max} \le s \le 0} \|\varphi(s)\|_\infty^v$, $\gamma = \max_{1 \le i \le n} \gamma_i$, *and* $\gamma_i \in (0,1)$ *is the unique positive solution of the equation*

$$\frac{f_i(v)}{v_i} + \frac{g_i(v)}{v_i}\gamma_i^{-d_{\max}} - \gamma_i = 0, \quad i = 1, 2, \cdots, n.$$

**Theorem 3.4. (Discrete-time case).** *Consider the system*

$$\begin{cases} x(k+1) = Ax(k) + Bx(k - d(k)), & k \in \mathbb{Z}_+ \\ x(s) = \varphi(s), & s \in \{-d_{\max}, \cdots, 0\} \end{cases} \tag{4}$$

*that satisfies A and B are nonnegative. The following statements are equivalent*:

i) *There exists a vector* $v > 0$ *such that*

$$(A + B)v < v.$$

ii) *System* (4) *is globally exponentially stable for all bounded time delays.*

## 3.2. The Case of Degree $\alpha > 1$

Inspired by the new techniques presented in [12], Dong continued the research for the case of homogeneous vector fields of degree greater than one and achieved some noteworthy results. Detailed proofs can be found in [13].

**Theorem 3.5 (Continuous-time case).** *Consider the system* (1) *that satisfies Assumption* 1 *and Assumption* 2. *The following statements are equivalent*:

i) *There exists a vector* $v > 0$ *such that*

$$f(v) + g(v) < \mathbf{0}.$$

ii) *System* (1) *is globally polynomially stable for all bounded time delays.*

*In addidion, if any of the two equivalent statements is satisfied, then*

$$\|x(t)\|_\infty^v \le \left(\|\varphi\|^{1-\alpha} + (\alpha - 1)\eta t\right)^{-\frac{1}{\alpha-1}}, \quad t \ge 0,$$

*where* $\|\varphi\| = \max_{-\tau_{\max} \le s \le 0} \|\varphi(s)\|_\infty^v$, $\eta \in \left(0, \min_{1 \le i \le n} \eta_i\right)$, *and* $\eta_i$ *is the unique positive solution of the equation*

$$\frac{f_i(v)}{v_i} + \frac{g_i(v)}{v_i}\left(1 + (\alpha - 1)\|\varphi\|^{\alpha-1}\tau_{\max}\eta_i\right) + \eta_i = 0, \quad i = 1, 2, \cdots, n.$$

**Theorem 3.6 (Discrete-time case).** *Consider the system* (2) *that satisfies Assumption* 1 *and Assumption* 2. *The following statements are equivalent*:

i) *There exists a vector* $v > 0$ *such that*

$$f(v) + g(v) < v.$$

ii) *System* (2) *is locally exponentially stable for all bounded time delays.*

*In addition, if any of the two equivalent statements is satisfied, then for any initial function satisfying* $\|\varphi\| < \gamma^{-1}$, *we have*

$$\|x(k)\|_\infty^v \le \gamma^{-1}\rho^{\alpha^{\left\lceil \frac{k}{d_{\max}+1} \right\rceil}}, \quad k \in \mathbb{Z}_+,$$

*where* $\rho = \|\varphi\|\gamma$, $\|\varphi\| = \max_{-d_{\max} \le s \le 0} \|\varphi(s)\|_\infty^v$, $\gamma = \max_{1 \le i \le n} \gamma_i$, *and* $\gamma_i \in (0,1)$ *is the* *unique positive solution of the equation*

$$\frac{f_i(v)}{v_i}+\frac{g_i(v)}{v_i}-\gamma_i^{\alpha-1}=0,\quad i=1,2,\cdots,n.$$

### 3.3. The Case of Degree $0<\alpha<1$

In [13], Dong presented two results for systems with no delay and constant delay. Subsequently, Xiao and colleagues extended Feyz's results in [11], improved some of Dong's findings in [13], and introduced new results for systems with time-varying bounded delays and homogeneous vector fields of degree less than one. Detailed proofs can be found in [14].

**Theorem 3.7 (Continuous-time case).** *Consider the delay-free positive system*

$$\begin{cases}\dot{x}(t)=f\left(x(t)\right)+g\left(x(t)\right), & t\geq 0\\ x(0)\in\mathbb{R}_+^n\end{cases}\tag{5}$$

*that satisfies Assumption* 1 *and Assumption* 2. *The following statements are equivalent*:

i) *There exists a vector* $v>0$ *such that*

$$f(v)+g(v)<\mathbf{0}.$$

ii) *System* (5) *is globally stable in finite time*.

*In addidion, if any of the two equivalent statements is satisfied, then the upper bound of the convergence time* $\mathbf{T}$ *is given by*

$$\mathbf{T}\leq\frac{\left(\left\|x(0)\right\|_\infty^v\right)^{1-\alpha}}{\eta(1-\alpha)},$$

*where* $\eta=\min_{1\leq i\leq n}\eta_i$ *with* $\eta_i$ *is the unique positive solution of the equation*

$$\frac{f_i(v)}{v_i}+\frac{g_i(v)}{v_i}+\eta_i=0,\quad i=1,2,\cdots,n.$$

**Theorem 3.8 (Continuous-time case).** *Consider the positive system with constant delay*

$$\begin{cases}\dot{x}(t)=f\left(x(t)\right)+g\left(x(t-\tau)\right), & t\geq 0\\ x(s)=\varphi(s), & s\in[-\tau,0]\end{cases}\tag{6}$$

*that satisfies Assumption* 1 *and Assumption* 2. *The following statements are equivalent*:

i) *There exists a vector* $v>0$ *such that*

$$f(v)+g(v)<\mathbf{0}.$$

ii) *System* (6) *is globally asymptotically stable for all* $\tau\geq 0$.

**Theorem 3.9 (Continuous-time case).** *Consider the system* (1) *that satisfies Assumption* 1 *and Assumption* 2 *and* $0<\alpha\leq 1$. *If there exists a vector* $v>0$

*such that*

$$f(v)+g(v)<\mathbf{0},$$

*then system* (1) *is globally exponentially stable for all bounded time delays. Specifically, each solution* $x(t)$ *of* (1) *satisfies*

$$\|x(t)\|_\infty^v \le \|\varphi\| e^{-\eta t}, \quad t \ge 0,$$

*where* $\|\varphi\| = \max_{-\tau_{\max}\le s\le 0} \|\varphi(s)\|_\infty^v$, $\eta \in \left(0, \min_{1\le i\le n} \eta_i\right)$, *and* $\eta_i$ *is the unique positive solution of the equation*

$$\frac{f_i(v)}{v_i} + \frac{g_i(v)}{v_i} e^{\eta_i \tau_{\max}} + \|\varphi\|^{1-\alpha} \eta_i = 0, \quad i = 1, 2, \cdots, n.$$

**Theorem 3.10 (Continuous-time case).** *Consider the system* (1) *that satisfies Assumption* 1 *and Assumption* 2. *If there exists a vector* $v > 0$ *such that*

$$f(v)+g(v)<\mathbf{0},$$

*then solution of system* (1) *is bounded in any fix time. Specifically, for any given constant* $\mathbf{T} > 0$, *the solution* $x(t)$ *satisfies*

$$\|x(t)\|_\infty^v \le \left(\|\varphi\|^{1-\alpha} - \eta t\right)^{\frac{1}{1-\alpha}},$$

*for all* $t \in (0, \mathbf{T})$ *and* $\|\varphi\| = \max_{-\tau_{\max}\le s\le 0} \|\varphi(s)\|_\infty^v$, $\eta \in \left(0, \min_{1\le i\le n} \eta_i\right)$, *where* $\eta_i \in \left(0, \frac{\|\varphi\|^{1-\alpha}}{\mathbf{T}}\right)$, $i = 1, 2, \cdots, n$ *is restricted by*

$$\frac{f_i(v)}{v_i^{\alpha(1-\alpha)}} + \frac{g_i(v)}{v_i^{\alpha(1-\alpha)}} \left(1 + \frac{\tau_{\max}\eta_i}{\|\varphi\|^{1-\alpha} - \mathbf{T}\eta_i}\right)^{\frac{\alpha}{1-\alpha}} + \frac{\eta_i}{1-\alpha} \le 0.$$

**Theorem 3.11. (Continuous-time case).** *Consider the system* (5) *that satisfies Assumption* 1 *and Assumption* 2. *If there exists a vector* $v > 0$ *such that*

$$f(v)+g(v)<\mathbf{0},$$

*then system* (5) *is finite-time stable. In particular, its solution* $x(t)$ *satisfies*

$$\begin{cases} \|x(t)\| \le \left(\left(\|x(0)\|_\infty^v\right)^{1-\alpha} - \eta t\right)^{\frac{1}{1-\alpha}}, & t < \frac{\left(\|x(0)\|_\infty^v\right)^{1-\alpha}}{\eta} \\ x(t) = \mathbf{0}, & t \ge \frac{\left(\|x(0)\|_\infty^v\right)^{1-\alpha}}{\eta} \end{cases}$$

*where* $\eta \in \left(0, \min_{1\le i\le n} \eta_i\right)$, *and* $\eta_i$ *is the unique positive solution of the equation*

$$\frac{f_i(v)}{v_i^{1-\alpha}} + \frac{g_i(v)}{v_i^{1-\alpha}} + \frac{1}{1-\alpha}\eta_i = 0, \quad i = 1, 2, \cdots, n.$$

Thus, the case of positive discrete-time systems with homogeneous vector fields of degree less than one has not been previously examined and studied by earlier authors.

## 4. Main Result

In this section, we provide a sufficient condition for global exponential stability for the case discrete-time of positive systems with homogeneous vector fields of degree less than one.

**Theorem 4.1 (Discrete-time case).** *Consider the system* (2) *that satisfies Assumption* 1 *and Assumption* 2 *and* $0<\alpha<1$. *If there exists a vector* $v>0$ *such that*

$$f(v)+g(v)<\mathbf{0},$$

*then system* (2) *is globally exponentially stable for all bounded time delays. Specifically, each solution* $x(t)$ *of* (2) *satisfies*

$$\|x(k)\|_{\infty}^{v}\le\|\varphi\|\gamma^{k},\quad k\in\mathbb{Z}_{+},$$

*where* $\|\varphi\|=\max_{-d_{\max}\le s\le 0}\|\varphi(s)\|_{\infty}^{v}$, $\gamma=\max_{1\le i\le n}\gamma_i$ *with* $\gamma_i$ *is the is the unique positive solution of the equation*

$$\frac{f_i(v)}{v_i}+\frac{g_i(v)}{v_i}\gamma_i^{-d_{\max}}-\|\varphi\|^{1-\alpha}\gamma_i=0,\quad i=1,2,\cdots,n.$$

*Proof.* For $\lambda\ge 1$, we observe that the function varies with the variable $\gamma_i$

$$h(\gamma_i)=f_i(v)+g_i(v)\gamma_i^{-d_{\max}}-v_i\|\varphi\|^{1-\alpha}\lambda^{1-\alpha}\gamma_i$$

which has $h(0)=f_i(v)>0$, due to the order-preserving property of $f(.)$. Furthermore, when $\gamma_i>0$ the function $h(.)$ is strictly decreasing. Therefore, there exists a unique $\gamma_i>0$ such that $h(\gamma_i)=0$.

Let $\gamma=\max_{1\le i\le n}\gamma_i$, then

$$f_i(v)+g_i(v)\gamma^{-d_{\max}}-v_i\|\varphi\|^{1-\alpha}\lambda^{1-\alpha}\gamma\le 0,\quad i=1,2,\cdots,n.$$

Define

$$z_i(k)=\frac{x_i(k)}{v_i}-\lambda\|\varphi\|\gamma^{k}.$$

We will prove $z_i(k)\le 0, k\in\mathbb{N}$ by induction on *k*. When $k=0$, according to the definition of $\|\varphi\|$, we have $\|x(k)\|_{\infty}^{v}\le\|\varphi\|\le\lambda\|\varphi\|$, with $\lambda\ge 1$. Therefore,

$$z_i(0)\le\|\varphi\|-\lambda\|\varphi\|=(1-\lambda)\|\varphi\|\le 0.$$

Assume that $z_i(k)\le 0$ holds for all $k\le m$, *i.e.*, $x(k)\le\lambda\|\varphi\|\gamma^{k}v$. Since $f,g$ are order-preserving and homogeneous of degree $\alpha>0$, we get

$$f(x(m))\le f(\lambda\|\varphi\|\gamma^{m}v)=\lambda^{\alpha}\|\varphi\|^{\alpha}\gamma^{m\alpha}f(v)$$

$$\begin{aligned}g(x(m-d(m)))&\le g\left(\lambda\|\varphi\|\gamma^{m-d(m)}v\right)\\&=\lambda^{\alpha}\|\varphi\|^{\alpha}\gamma^{(m-d(m))\alpha}g(v)\\&\le\lambda^{\alpha}\|\varphi\|^{\alpha}\gamma^{(m-d_{\max})\alpha}g(v)\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{v_i}x_i(m+1) &= \frac{1}{v_i}\left[f_i(x(m))+g_i(x(m-d(m)))\right]\\ &\le \frac{1}{v_i}\left[\lambda^\alpha\|\varphi\|^\alpha\gamma^{m\alpha}f_i(v)+\lambda^\alpha\|\varphi\|^\alpha\gamma^{(m-d_{\max})\alpha}g_i(v)\right]\\ &= \frac{1}{v_i}\lambda^\alpha\|\varphi\|^\alpha\gamma^{m\alpha}\left[f_i(v)+g_i(v)\gamma^{-d_{\max}}\right]\\ &\le \frac{1}{v_i}\lambda^\alpha\|\varphi\|^\alpha\gamma^{m\alpha}v_i\|\varphi\|^{1-\alpha}\lambda^{1-\alpha}\gamma\\ &= \lambda\|\varphi\|\gamma^{m\alpha+1}\end{aligned}$$

Since $0<\gamma<1$ then

$$z_i(m+1)=\frac{x_i(m+1)}{v_i}-\lambda\|\varphi\|\le\lambda\|\varphi\|\gamma^{m\alpha+1}-\lambda\|\varphi\|=\lambda\|\varphi\|\left(\gamma^{m\alpha+1}-1\right)\le 0,$$

*i.e.*, $z_i(m+1)\le 0$ holds. Thus $\|x(k)\|_\infty^v\le\lambda\|\varphi\|\gamma^k$, $\forall k\in\mathbb{Z}_+$. Let $\lambda\to 1^+$, we obtain a true statement. □

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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